https://www.youtube.com/watch?v=P8tXDuP9t88&list=PL08ef9eJxtJa3svcoUCDmG-_lDx2ihWKF
أفضل شرح للماده

ال Matrix
عباره عن rectangular وكل رقم فيها يسمى entries والعدد اللي نضربه بال matrix كلها اسمه scalar

ملاحظه : عند ((جمع وطرح)) ال matrix لازم تكون من نفس الحجم اما عند ((الضرب)) لازم يكون عدد أعمدة المصفوفه الأولى يساوي عدد صفوف المصفوفه الثانيه .

مثال على ((الضرب)) لمصفوفه من حجم 3(row)×4 2×3(column) :

ينتج مصفوفه 2×4

عملية الضرب تكون صف × عمود

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 4} \\ 2 \quad 6 \quad 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$(1.4) + (2.0) + (4.2) = 12$$
$$(1.1) - (2.1) + (4.7) = 27$$
$$(1.4) + (2.3) + (4.5) = 30$$
$$(2.4) + (6.0) + (0.2) = 8$$
$$(2.1) - (6.1) + (0.7) = -4$$
$$(2.3) + (6.1) + (0.2) = 12$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

خصائص ال matrix

$AB \neq BA$ لأن عملية الضرب ليست ابداليه Not comutitive

$$A - A = A + (-A) = 0$$

$$A - B = -B - A$$

$$A - 0 = A$$

$$A + B = B + A$$

$$A + (B + C) = (B + A) + C$$

$$A(BC) = (AB)C$$

ال matrix اربع أنواع :

1- Zero matrix: هي مصفوفه من أي حجم كلها اصفار مثل :

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

2- Identity matrix : هي Special Matrix مصفوفه مربعه وكلها اصفار ماعدا قطرها واحدات مثل :

A square matrix with 1's on the main diagonal and zeros elsewhere is called an *identity matrix*. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3- Diagonal matrix : هي مثل ال Identity matrix ولكن الفرق انها ليست Special وقطرها ارقام مثل :

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

4- **Triangular matrix** : مصفوفه مربعه وفيها مثلث صفري والأرقام تكون فوق أوتحت :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

العلميات على ال **matrix** :

1- **Trace matrix** : مصفوفه مربعه اذا طلب بالسؤال نطلع ال **Trace matrix** فقط نجمع اعداد قطرها ال **main daiognal** مثل :

DEFINITION 8 If A is a square matrix, then the *trace of* A, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

EXAMPLE 11 Trace of a Matrix

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\mathrm{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \mathrm{tr}(B) = -1 + 5 + 7 + 0 = 11$$

2-Determinants مصفوفه مربعه واذا طلب بالسؤال ال Determinants لها حالتين :

-

▶ EXAMPLE 7 A Technique for Evaluating 2 x 2 and 3 x 3 Determinants

1

مصفوفه 2×2 اضرب بطريقة المقص وبينهم طرح

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

2

مصفوفه مربعه اكبر من 2×2 مثل 3×3 واكبر استخدم هالطريقه

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right|$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$ ◀

ملاحظه : اقواس ال Determinants مثل اقواس القيمه المطلقه بدون رؤوس .

3- Determinants Cofactor :

قانونها :

$C_{ij} = (-1)^{i+j} \times Mij$ ال Mij تدل على minor وال C_{ij} وال i row وال j column

(والسؤال عليها يجي بطريقتين) :

EXAMPLE 1 Finding Minors and Cofactors

Let

1

اذا ماحدد بالسؤال صف او عمود معين
الخيار مفتوح ((وافضل خيار اني
اختار الصف اللي فيه الرقم (0) لأن
مايحتاج نعمل له Determinants))

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

نحذف الصف الأول العمود

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

ناتج ال Cofactor دائما
نفس ناتج ال Mainor
بأختلاف الأشاره اذا كان
حاصل جمع الأسس زوجي
ينزل زي ماهو .. أما اذا
كان حاصل جمع الأسس
فردي فقط ننزل العدد زي
ماهو و نعكس الأشاره

The cofactor of a_{11} is

ينزل زي ماهو لأن جمع الأسس عدد زوجي

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

Similarly, the minor of entry a_{32} is

نحذف الصف الثالث العمود الثاني

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

فقط نعكس الأشاره لأن ناتج جمع الأسس عدد فردي

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

▶ EXAMPLE 3 Cofactor Expansion Along the First Row

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

▶ EXAMPLE 4 Cofactor Expansion Along the First Column

Let A be the matrix in Example 3, and evaluate $\det(A)$ by cofactor expansion along the first column of A.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

This agrees with the result obtained in Example 3.

ملاحظه : الأشارات تتبع هذي الطريقه

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

ملاحظه :

THEOREM 2.2.1

Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

هذا مثال على اثبات نظريه :

EXAMPLE 1 $\det(A + B) \neq \det(A) + \det(B)$

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A + B) = 23$; thus

$$\det(A + B) \neq \det(A) + \det(B)$$

4-Symmetric matrix : مصفوفه مربعه وهي Special ومساويه لل Transpose matrix :

DEFINITION 1 A square matrix A is said to be *symmetric* if $A = A^T$.

Properties of Symmetric matrix :

THEOREM 7.2.2 *If A is a symmetric matrix, then:*

(a) *The eigenvalues of A are all real numbers.*

(b) *Eigenvectors from different eigenspaces are orthogonal.*

5- Transpose matrix : مصفوفه من أي حجم واذا طلب بالسؤال نطلع ال Transpose matrix

((لها حالتين)) :

1 اذا كانت المصفوفه مربعه نثبت قطرها ونبدل مكان ال entries اللى باليمين يصير باليسار والعكس مثل هذى مصفوفه 3×3 :

$$A=\begin{bmatrix}1 & -2 & 4\\ 3 & 7 & 0\\ -5 & 8 & 6\end{bmatrix} \rightarrow \begin{bmatrix}1 & -2 & 4\\ 3 & 7 & 0\\ -5 & 8 & 6\end{bmatrix} \rightarrow A^T=\begin{bmatrix}1 & 3 & -5\\ -2 & 7 & 8\\ 4 & 0 & 6\end{bmatrix}$$

Interchange entries that are symmetrically positioned about the main diagonal.

2 اذا كانت المصفوفه غير مربعه فقط نقلب مكان الصفوف الى اعمده مثل هذي مصفوفه 2×3 تصير بعد القلب 3×2 :

$$\begin{bmatrix}4 & 5 & 7\\ 2 & 6 & 3\end{bmatrix}$$

$$\begin{bmatrix}4 & 2\\ 5 & 6\\ 7 & 3\end{bmatrix}$$

Transpose Matrix Properties:

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

مثل :

$$(AB)^T = B^T A^T$$

من خصائصها أيضاً لو أسوي Transpose لنفس ال Matrix مرتين راح ترجع لحالتها الأصليه مثل :

$$(A^T)^T = A$$

من خصائصها أيضاً في الطرح لازم تكون نفس الترتيب يعني ليست ابداليه مثل :

$$(A - B)^T = A^T - B^T$$

: System أنواع ال

1- **Consistent** : at least one solution.

2- **Inconsistent** : has no solution.

3- **Homogeneous Systems**: all equations are set =0

مثال على **Homogeneous** :

$$3x + 2y = 0$$

مثال على **non Homogeneous** :

$$3x + 2y = 5$$

ملاحظه: عشان المعادله تكون (**Linear equations**) لازم تكون :

have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions , Variables appear only to the first power.

مثال :

- Ex 1: (Linear or Nonlinear)

Linear $(a)\ 3x + 2y = 7$

$(b)\ \frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear $(c)\ x_1 - 2x_2 + 10x_3 + x_4 = 0$

$(d)\ (\sin\frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

Nonlinear $(e)\ xy + z = 2$

not the first power

Exponential

$(f)\ e^x - 2y = 4$ Nonlinear

Nonlinear $(g)\ \sin x_1 + 2x_2 - 3x_3 = 0$

trigonometric functions

$(h)\ \frac{1}{x} + \frac{1}{y} = 4$ Nonlinear

not the first power

Solving Linear System by :

1-Gauss Elimination

2-Gauss-Jordan Elimination

3-Inverse Matrix Method

4-Cramer Rule

1-Row-echelon form (Gaussian elimination: is the procedure for reducing a matrix to a row-echelon form).

من شروطها ان الصف الأول يبدأ بالرقم 1 وقطرها واحدات وماتكون الواحدات فوق بعض وتحتوي على مثلث صفري اسفل اليسار مثل :

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced row-echelon form (Gauss-Jordan elimination is the procedure for reducing a matrix to a reduced row-echelon form).

من شروطها قطرها واحدات تحتوي على مثلث صفري والرقم 1 يكون فوقه وتحته اصفار مثل :

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ملاحظه :

ال leading 1 يكون فوقه وتحته اصفار .

ملاحظه :

1-row-echelon form not unique كل طالب اجابته مختلفه عن الآخر

2-reduced row echelon form unique إجابات الطلاب تكون موحده

Example 1 مقتبس من ملخص(EMS) :

Use Gaussian Elimination to Solve the system of linear equation?

$x - 3y + z = 4$

$2x - 8y + 8z = -2$

$-6x - 3y - 15z = 9$

الحل :

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & -3 & -15 & 9 \end{bmatrix}$$

نضرب الصف الأول بـ سالب -2 ونضيفه للصف الثاني

$R_2 \to R_2 - 2R_1$

نضرب الصف الأول بـ موجب +6 ونضيفه للصف الثالث

$R_3 \to R_3 + 6R_1$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{bmatrix}$$

نقسمه الصف الثاني على 2 −

$R_2 \to \frac{R_2}{-2}$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & -15 & -9 & 33 \end{bmatrix}$$

نضرب الصف الثاني بـ موجب +15 ونضيفه للصف الثالث

$R_3 \to R_3 + 15R_2$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & -54 & 108 \end{bmatrix}$$

نقسمه الصف الثاني على 54 −

$R_3 \to \frac{R_3}{-54}$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\boxed{z = -2}$	$y - 3z = 5$ $y - 3(-2) = 5$ $y + 6 = 5$ $y = 5 - 6$ $\boxed{y = -1}$	$x - 3y + z = 4$ $x - 3(-1) + (-2) = 4$ $x + 3 - 2 = 4$ $x + 1 = 4$ $x = 4 - 1$ $\boxed{x = 3}$

Example2 :

Solving the system by using elementary operation أو Gaussian elimination?

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

الهدف هنا نخلي القطر كله واحدات والمثلث الصفري يكون تحت اسفل اليسار

طريقة كتابة ال system على شكل
أي على شكل مصفوفه Augmented

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} (1)r_1 + r_2 \to r_2$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} (-2)r_1 + r_3 \to r_3$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} (1)r_2 + r_3 \to r_3$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} (\frac{1}{2})r_3 \to r_3$$

ملاحظه : أفضل تكنيك اذا كان الرقم اللي ابي احوله ل 0 أو 1 بالعمود الأول اضربه أو اجمعه أو اطرحه من الصف الأول وهكذا العمود الثاني بالصف الثاني وافضل تكنيك عشان احول الرقم 2 الى 1 اضربه ب $\frac{1}{2}$ اما اذا ابي احول الرقم 3 الى 1 اضربه ب $\frac{1}{3}$ وهكذا

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

x = 1

y = -1

z = 2

ملاحظه : هذا الناتج طلع من التعويض

اذا ضربنا المصفوفه في مقلوبها يعطينا مصفوفة الوحده

: Inverse Matrices

1 $A^{-1}A$ = 1 2 AA^{-1} = 1 3 AB = BA = 1

من شروطها تكون مربعه والهدف منها اسوي مثلث صفري فوق وتحت وقطرها كله واحدات وقانونها اذا كان حجمها 2 × 2

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

قانونها اذا كانت اكبر من 2 × 2

$$(AB)^{-1} = B^{-1}A^{-1}$$

قانونها ب Transpose Inverse

$$(A^T)^{-1} = (A^{-1})^T$$

قانونها ب Determinants Inverse

$$\det(A^{-1}) = \frac{1}{det\ (A)}$$

مثال :

Solve the matrix Finding A^{-1}

Calculating the Inverse of a 2 × 2 Matrix :

$$A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$$

$$det(A) = (6)(2) - (1)(5)$$

$$= 12 - 5 = 6$$

اولآ : نوجد القيمه المحدده

ثانيا : نوجد المقلوب حسب القانون

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{5}{6} & 1 \end{bmatrix}$$

Using the Row Operator (find invers in new method) : بطريقه أخرى

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$$

$$\left[A \mid I\right] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right]$$

مهم نكتب هالخطوه (ضرب المصفوفه في مصفوفة الوحده

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right] \qquad \boxed{2R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right] \qquad \boxed{(-1)R_2 \to R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \qquad \boxed{3R_2 - R_1 \to R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \textit{and } A^{-1}\begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$$

مثال آخر على 3 × 3 matrix :

Solve the matrix Finding A^{-1}

الخطوه هذي لازم اكتبها مثل حجم المصفوفه المطلوبه بالسؤال لأن الهدف احول الطرف الأيسر مثل الطرف الأيمن

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \leftarrow$$ We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \leftarrow$$ We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow$$ We multiplied the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow$$ We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow$$ We added -2 times the second row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

namely, $\mathbf{x} = A^{-1}\mathbf{b}$

مثال على هذا القانون :

EXAMPLE 1 Solution of a Linear System Using A^{-1}

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

اول خطوه اكتبها على شكل مصفوفه بهذا الشكل

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

ثاني خطوه اطلع ال inverse

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

ثالث خطوه اضرب ال inverse بالمصفوفه b

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1, x_2 = -1, x_3 = 2$.

Inverse Transformations :

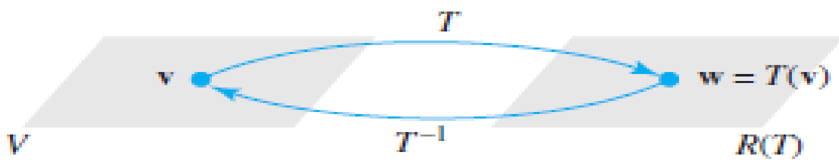
$$T^{-1}(T(\mathbf{v})) = T^{-1}(\mathbf{w}) = \mathbf{v}$$

$$T(T^{-1}(\mathbf{w})) = T(\mathbf{v}) = \mathbf{w}$$

THEOREM 8.3.2 *If $T_1: U \to V$ and $T_2: V \to W$ are one-to-one linear transformations, then*

(a) $T_2 \circ T_1$ *is one-to-one.*

(b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.



مثال :

EXAMPLE 4 An Inverse Transformation

Let $T: R^3 \to R^3$ be the linear operator defined by the formula

$$T(x_1, x_2, x_3) = (3x_1 + x_2,\ -2x_1 - 4x_2 + 3x_3,\ 5x_1 + 4x_2 - 2x_3)$$

Determine whether T is one-to-one; if so, find $T^{-1}(x_1, x_2, x_3)$.

Solution It follows from Formula 12 of Section 4.9 that the standard matrix for T is

$$[T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

(verify). This matrix is invertible, and from Formula 7 of Section 4.10 the standard matrix for T^{-1} is

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

Expressing this result in horizontal notation yields

$$T^{-1}(x_1, x_2, x_3) = (4x_1 - 2x_2 - 3x_3,\ -11x_1 + 6x_2 + 9x_3,\ -12x_1 + 7x_2 + 10x_3)$$

Cramer's Rule

$$x_1 = \frac{det(A_1)}{det(A)} \quad , \quad x_2 = \frac{det(A_2)}{det(A)} \quad , \quad x_3 = \frac{det(A_3)}{det(A)}$$

مثال:

EXAMPLE 8 Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve

ملاحظه ال System عباره عن معادله فيها مجاهيل

$$\begin{aligned} x_1 + + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Solution space of the homogeneous system of equations Ax = 0 Called null Space.

THEOREM 4.7.1 *A system of linear equations* $A\mathbf{x} = \mathbf{b}$ *is consistent if and only if* $\mathbf{b}$ *is in the column space of* A.

Augmented Matrix :

مثال :

$$x - 4y + 3z = 5$$

$$-x + 3y - z = -3$$

$$2x \qquad - 4z = 6$$

Augmented Matrix :

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$$

Coefficient Matrix:

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

خطوات حل ال **least square solution of linear matrix** :

1. Express the system in matrix :

$$Ax = b$$

2. Best Approximation أو Least Squares solutions :

$$A^T Ax = A^T b$$

3. Error vectors :

$$b - Ax$$

4. Erorr :

$$||b - Ax||$$

مثال عليها :

EXAMPLE 1 Least Squares Solution

(a) Find all least squares solutions of the linear system

$$\begin{aligned} x_1 - x_2 &= 4 \\ 3x_1 + 2x_2 &= 1 \\ -2x_1 + 4x_2 &= 3 \end{aligned}$$

(b) Find the error vector and the error.

Solution

(a) It will be convenient to express the system in the matrix form $A\mathbf{x} = \mathbf{b}$, where

1

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

It follows that

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is

2

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Solving this system yields a unique least squares solution, namely,

$$x_1 = \frac{17}{95}, \quad x_2 = \frac{143}{285}$$

(b) The error vector is

3

$$\mathbf{b} - A\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{95}{57} \end{bmatrix} = \begin{bmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{bmatrix}$$

and the error is

4

$$\|\mathbf{b} - A\mathbf{x}\| \approx 4.556$$

ال Vectors in coordinate system نوعين :

1- Gumatrie

2- coordinate system

ملاحظه: ال vector عند جمعهم او طرحهم او ضربهم لازم يكون من نفس ال space

مثال على two space vector تحتوي على 2 component :

V=(1,4)

مثال على vector three space يحتوي على 3 component :

V=(5,7,6)

امثله على ال

column vector or column matrix is an $m \times 1$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

row vector or row matrix is a $1 \times m$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix}.$$

ال Vector Norm له ثلاث أسماء :

norm - magnitude - Length ورمزها $\|v\|$ وقانونها :

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 \dots\dots + v_n^2}$$

قانون ال Unit Vectors :

$$u = \frac{1}{\|v\|} v$$

ال Euclidean norm قانونها :

$$\|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$$

ال Distance vector قانونها :

$$d(u, v) = \|u - v\|$$

مثال سؤال على ال Norm :

If $v = (-3, 2, 1)$, find $||v||$ ؟؟

الجواب : مجرد تعويض في هذ القانون : $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 \dots \dots . + v_n^2}$

$\|v\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$

لازم اكتب القوسين اذا كان العدد سالب لان بدونه راح يعطيني ناتج مختلف بالآله الحاسبه

If $v = (1, -3, 2)$, and $w = (4, 2, 1)$ find $v + w$ and $v - w$??

$v + w = (1 + 4\ , -3 + 2\ , 2 + 1)$

$= (5, -1, 3)$

$v - w = (1 - 4\ , -3 - 2\ , 2 - 1)$

$= (-3\ , -5\ , 1)$

مثال سؤال على ال Distance vector :

Find the distance between $u = (0, 2)$ and $v = (2, 0)$??

$d(u, v) = \|u - v\| = (0 - 2), (2 - 0) = \|-2, 2\|$

$= \sqrt{(-2)^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8}$

ال Dot prouduct او ال inner prouduct او ال Euclidean inner prouduct او ال Euclidean Dot prouduct قانونها :

ناتجها دائما عدد scalar

$$u \,.\, v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

ال complex Euclidean inner prouduct او complex Dot prouduc قانونها :

$$u \,.\, v = u_1 \overline{v_1} + u_2 \overline{v_2} + \dots + u_n \overline{v_n}$$

Inner Products :

DEFINITION 1 An *inner product* on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a *real inner product space*.

قانونها :

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Properties of the Dot Product :

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

Properties of Euclidean Dot prouduct :

THEOREM 5.3.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* C^n, *and if* k *is a scalar, then the complex Euclidean inner product has the following properties:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$ **[Antisymmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{u} \cdot k\mathbf{v} = \overline{k}(\mathbf{u} \cdot \mathbf{v})$ **[Antihomogeneity property]**

(*e*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$. **[Positivity property]**

Properties of Inner Products :

THEOREM 6.1.2 *If* **u**, **v**, *and* **w** *are vectors in a real inner product space V, and if k is a scalar, then:*

(a) $\langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$

(b) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$

(c) $\langle \mathbf{u}, \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{w}\rangle$

(d) $\langle \mathbf{u}-\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{w}\rangle$

(e) $k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, k\mathbf{v}\rangle$

مثال على Dot prouduct او inner prouduct :

السؤال : If u = (3, −2) and v = (4, 5) ؟؟

الجواب : $u \cdot v = (3)(4) + (-2)(5) = 12 - 10 = 2$

مثال على Dot Products and Matrices :

Table 1

Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}^T\mathbf{v} = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u} = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a row matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\mathbf{v} = \mathbf{v}^T\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}\mathbf{v} = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u}^T = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a column matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\mathbf{u} = \mathbf{u}^T\mathbf{v}^T$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{v}\mathbf{u} = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$ $\mathbf{u}^T\mathbf{v}^T = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$
u a row matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{u}\mathbf{v}^T = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}\mathbf{u}^T = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$

قانون ال cos Ø اذا كانت angle بين nonzero vector وهم u , v :

$$\cos \emptyset = \frac{u.v}{\|u\|\|v\|}$$

angle between u and v :

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

DEFINITION 1 Two vectors **u** and **v** in an inner product space are called ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

مثال على ال cos Ø اذا كانت angle بين nonzero vector :

السؤال : Find the angle between the vectors u = (2, 5) and v = (4, −3) ??

الجواب :

$$\cos \emptyset = \frac{u.v}{\|u\|\|v\|} = \frac{(2)(4)+(5)(-3)}{\sqrt{2^2+5^2}+\sqrt{4^2+3^2}} = \frac{8-15}{\sqrt{4+25}\sqrt{16+9}} = \frac{-7}{\sqrt{29}\sqrt{25}} = -\ \frac{7}{5\sqrt{29}}$$

$$\emptyset = \cos^{-1}\left(-\frac{7}{5\sqrt{29}}\right) = 105.1$$

ال cross product او ال outer product قانونها :

ناتجها دائما vector وليس scalar

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

القانون الأول

or, in determinant notation,

هنا احذف الأول

هنا احذف الثاني

هنا احذف الثالث

اختار واحد من القانونين بالحل

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \quad (1)$$

القانون الثاني

ملاحظه : الأشارات تتبع هذي الطريقه

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

الحاله الشاذه في ال cross product عملية الضرب ابداليه بهذي الطريقه

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

$\mathbf{u} \times \mathbf{u} = 0$

العلاقه بين ال cross product وال Dot prouduct :

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (*Lagrange's identity*)

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (*relationship between cross and dot products*)

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ (*relationship between cross and dot products*)

ال Subspace :

لها شرطين :

1-if u and v re vectors in W , then u + v is in W.

2-if k any scalar and u is any vectors in W , then k u is in W.

ملاحظه :

شروط ال vector space هنا ذكرت شرطين فقط باقي الشروط مكرره مثل أي شي نجمعه مع ال 0 بيعطيني العدد نفسه واي عدد نجمعه مع نظيره الجمعي بيعطيني 0 .... الخ :

1-if u and v are objects in V then u + v is V

2-if k is any scalar and u is any object in V then k u is in V.

مثال على ال Row Space :

Given the 3 × 4 matrix. Find the basis for Row space of A and its dimension. Where

اشتغل على الصف الأول لأنه المطلوب بالسؤال

$$A=\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

المصفوفه غير مربعه اخذ فقط الجزء المربع منها واسوي لها row echlone form (اسوي مثلث صفري بالأسفل جهة اليسار وقطرها واحداث)

As performing elementary row operation do not change the row space. We convert A to reduced row echelon form.

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the basis is (1,4,5,2) $(0,1,1,\frac{4}{7})$ and the dimension of the row space of A is 2.

شروط ال Linear dependent :

THEOREM 4.3.2

ال vector اللي يحتوي على اكثر من عنصر ومن ضمنها رقم 0

3 *A finite set that contains* **0** *is linearly dependent.*

1 *A set with exactly one vector is linearly independent if and only if that vector is not* **0**.

2 *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

يوجد فيها scalar لعدد آخر

اذا ماتوفرت هالشروط راح تكون

Linear independent

نلاحظ أن ناتج ال Linear dependent لايساوي 0 اما ناتج ال Linear independent يساوي 0

مثال على ال Linear dependent مقتبس من ملخص EMS :

Determine whether the vectors the set $S\{(1,-2,3),(5,6,-1),(3,2,1)\}$ is L.I or L.D ?

Solution الحل

$R^3 \quad i=(1,-2,3) \quad j=(5,6,-1) \quad k=(3,2,1)$

$C_1 i + C_2 j + C_3 k = 0$ بالتعويض في المعادلة

$C_1(1,-2,3) + C_2(5,6,-1) + C_3(3,2,1) = (0,0,0)$ نضرب

$(C_1, -2C_1, 3C_1) + (5C_2, 6C_2, -1C_2) + (3C_3, 2C_3, C_3) = (0,0,0)$ نجمع

$(C_1 + 5C_2 + 3C_3 \;,\; -2C_1 + 6C_2 + 2C_3 \;,\; 3C_1 - 1C_2 + C_3) = (0,0,0)$

$C_1 + 5C_2 + 3C_3 = 0$

$-2C_1 + 6C_2 + 2C_3 = 0$

$3C_1 - 1C_2 + C_3 = 0$

لو قمنا بحل المعادلات فإن النتيجة تكون صفر لجميع المتغيرات

$\boxed{C_1 = C_2 = C_3 = 0 \quad \text{then L.I}}$

Span :

Subset of vectors space **V** that is formed from all possible linear combinations of the vectors in a nonempty set **S** is called Span of **S.**

Transformation maps denote $f : V \rightarrow W$

Special case where **V = W** called operator

مثال على ال **Span** :

Example 3 Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution The first step is to find the general solution of $Ax = 0$ in terms of free variables.

After transforming the augmented matrix $[A \quad 0]$ to the reduced row echelon form and we get;

هذي الأصفار لازم نكتبها

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

بدل الصف الأول بالثاني عشان الرقم 1 يكون الأول و نتبع طريقة **row echlon form**

which corresponds to the system

$$\begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

اكتب الناتج على شكل **system**

The general solution is

$x_1 = 2x_2 + x_4 - 3x_5$

$x_2 = \textit{free variable}$

$x_3 = -2x_4 + 2x_5$

$x_4 = \textit{free variable}$

$x_5 = \textit{free variable}$

نكتب المعادله مره ثانيه ال **leading 1** بيكون على الطرف الأيسر واكتب باقي المعادله على الطرف الأيمن واللي يكون **ireading** ولايوجد له معادله نكتب قدامه هالجمله

Next, decompose the vector giving the general solution into a linear combination of vectors where the weights are the free variables. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\substack{\uparrow \\ u}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\substack{\uparrow \\ w}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

$$= x_2 \boldsymbol{u} + x_4 \boldsymbol{v} + x_5 \boldsymbol{w} \quad (3)$$

نحول الطرف الايسر والأيمن الى vector matrix

Every linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is an element of Nul A. Thus $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a spanning set for Nul A.

Vector Basis :

شروطها:

1-Lineary independent.

2-Span.

مثال :

Example 5 Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{v_1, v_2, v_3\}$ is a basis for R^3.

Solution Since there are exactly three vectors here in R^3, we can use one of any methods to determine whether they are basis for $\mathbb{R}^3$ or not. For this, let solve with help of matrices. First form a matrix of vectors i.e. matrix $A = [v_1 \quad v_2 \quad v_3]$. If this matrix is invertible (i.e. $|A| \neq 0$ determinant should be non zero).

For instance, a simple computation shows that $det\ A = 6 \neq 0$. Thus A is invertible. As in example 3, the columns of A form a basis for R^3.

Example 7 Check whether the set of vectors $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$ is basis for R^3?

Now $\quad \det \begin{bmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{bmatrix} = 2(8) - 4(4) + 0 = 0$

Therefore, the system (A) is inconsistent, and, consequently, the set S does not span the space V.

Orthogonal :

شروطها :

1. Perpendicular
2. u . v = 0

orthogonal set of unit vector is called an orthogonal set

Orthogonal Basis :

قانونها :

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\|u\| \sqrt{u_1^2 + u_2^2 + u_3^2 \dots \dots . + u_n^2}$$

$$\mathbf{d}\,(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Orthogonal Matrices :

DEFINITION 1 A square matrix A is said to be *orthogonal* if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I \tag{1}$$

Properties of Orthogonal Matrices :

THEOREM 7.1.2

(a) $\boxed{\text{The inverse of an orthogonal matrix is orthogonal.}}$

(b) *A product of orthogonal matrices is orthogonal.*

(c) $\boxed{\text{If } A \text{ is orthogonal, then } \det(A) = 1 \text{ or } \det(A) = -1.}$

ممكن تجي بهذي الطريقه :

If A and B are Orthogonal matrices of the same size

شرح Orthogonal Matrix

قانون
$A^{-1} = A^{T}$
$A^{T}A = I$

Show that matrix $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ **is orthogonal.**

الحل:

نطبق القانون لاثبات أن المصفوفة متعامدة :

$$A^{T}A = I$$

أولا – نجد تبديل المصفوفة (transpose matrix)

$$A^{T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

ثانياً – نضرب المصفوفة في تبديل المصفوفة (transpose matrix)

$$AA^{T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \left(\left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right)\right) & \left(\left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)\right) \\ \left(\left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)\right) & \left(\left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right)\right) & \left(\left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)\right) \\ \left(\left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right)\right) & \left(\left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$AA^{T} = I$ **is Orthogonal**

Find A^{-1} ?

بما أن $A^{-1} = A^{T}$ فإن الجواب هو تبديل المصفوفة A^{T} هو :

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Orthogonal Diagonalization :

Similar to :

$$P^T AP = B$$

A is Orthogonal diagonalizable and Orthogonal set of n eigenvectors and Symmetric

مثال :

EXAMPLE 1 A 3 × 3 Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

مثال على orthogonal :

السؤال :

Determine whether the vectors in each pair are perpendicular

u = (3, 5) and v = (2, −8)
u = (2, 1) and v = (−1, 2) ??

الجواب :

u · v = (3)(2) + (5)(−8) = 6 − 40 = −34 ≠ 0

so u and v are not perpendicular.

u · v = (2)(−1) + (1)(2) = −2 + 2 = 0

so u and v are perpendicular

Dimension :

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

مثال على ال **Dimension :**

Exercises

For each subspace in exercises 1-6, (a) find a basis and (b) state the dimension.

1. $\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \boldsymbol{R} \right\}$

t = [-2 , 1 , 3]

s = [1, 1, 0]

عطاني 2 vector pace
معناها 2 Dimension

2. $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \text{ in } \boldsymbol{R} \right\}$

a = [0 , 1 , 0 , 1]

b = [0 , -1 , 1 , 2]

c = [2 , 6 , -3 , 0]

عطاني 3 vector pace
معناها 3 Dimension

مهم

THEOREM 4.5.6 *If W is a subspace of a finite-dimensional vector space V, then:*

(a) *W is finite-dimensional.*

(b) $\dim(W) \leq \dim(V)$.

(c) *$W = V$ if and only if* $\dim(W) = \dim(V)$.

coefficient of the linear combination expressed in the form :

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$$

Linear combination :

Find a linear combination let $u = (1,2,-1)$ and $v = (6,4,2) \in R^3$, $show\ that$

$w = (9,2,7)$ is linear combination of u & v.

Solution الحل

المعطيات في السؤال

$w = (9,2,7)$, $u = (1,2,-1)$, $v = (6,4,2)$

بالتعويض في المعادلة

$w = a\ u + b\ v$ 1

$(9,2,7) = a\ (1,2,-1) + b\ (6,4,2)$ ضرب

$= (a, 2a, -a) + (6b, 4b, 2b)$

$= (a + 6b\ ,\ 2a + 4b\ ,\ -a + 2b)$

نحول العناصر الى معادلات

$a + 6b = 9$ 2

$2a + 4b = 2$ 3

$-a + 2b = 7$ 4

نجمع المعادلة 2 مع المعادلة 4

$\cancel{a} - \cancel{a} + 6b + 2b = 9 + 7$

$8b = 16$

$\boxed{b = \frac{16}{8} = 2}$

بالتعويض في المعادلة 2

$a + 6(2) = 9$

$a + 12 = 9$

$a = 9 - 12$

$\boxed{a = -3}$

بالتعويض في المعادلة 1

$w = a\ u + b\ v$

$\boxed{w = -3\ u + 2\ v}$

الناتج النهائي

(Rank < variables >) dimension of the row space and column space of matrix.

(Nullity < parameters >) dimension of the null space of A.

Rank (A) + nullity (A) = n

DEFINITION 3 Let $T: V \to W$ be a linear transformation. If the range of T is finite-dimensional, then its dimension is called the ***rank of T***; and if the kernel of T is finite-dimensional, then its dimension is called the ***nullity of T***. The rank of T is denoted by rank(T) and the nullity of T by nullity(T).

مثال على ال Rank : (غالبا تجي خيارات ويكون السؤال عليها matrix حجمها 6 × 5 ومطلوب نطلع ال rank او ال nullity فقط اطرح عدد الأعمده من المعطى بالسؤال سواء rank او nullity)

مثال :

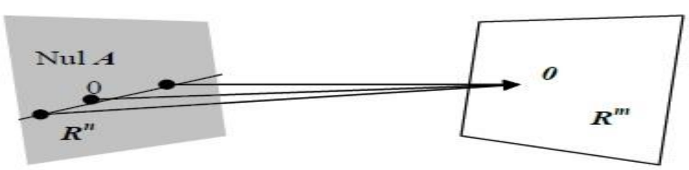


Figure 1

Example 1 Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and let $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $u \in \text{Nul } A$.

Solution To test if u satisfies $Au = 0$, simply compute

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ Thus u is in Nul A.

Image, Kernel and Range :

Image of any vector v in V expressed as :

$$T(v) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$$

THEOREM 8.1.3 *If $T: V \to W$ is a linear transformation, then:*

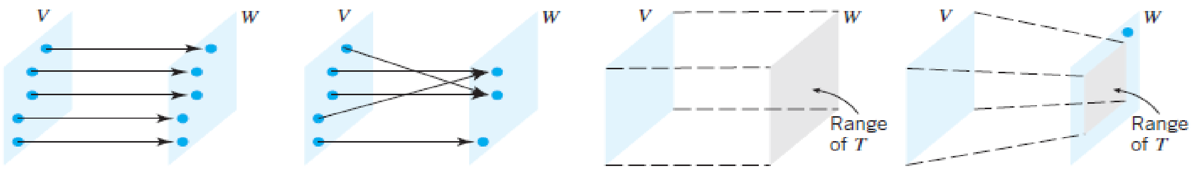
(a) *The kernel of T is a subspace of V.*

(b) *The range of T is a subspace of W.*

somorphism :

DEFINITION 1 If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be ***one-to-one*** if T maps distinct vectors in V into distinct vectors in W.

DEFINITION 2 If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be ***onto*** (or ***onto*** W) if every vector in W is the image of at least one vector in V.



One-to-one. Distinct vectors in V have distinct images in W.

Not one-to-one. There exist distinct vectors in V with the same image.

Onto W. Every vector in W is the image of some vector in V.

Not onto W. Not every vector in W is the image of some vector in V.

THEOREM 8.2.2 *If V is a finite-dimensional vector space, and if $T: V \to V$ is a linear operator, then the following statements are equivalent.*

(*a*) *T is one-to-one.*

(*b*) $\ker(T) = \{\mathbf{0}\}$.

(*c*) *T is onto* [*i.e.*, $R(T) = V$].

DEFINITION 3 If a linear transformation $T: V \to W$ is both one-to-one and onto, then T is said to be an ***isomorphism***, and the vector spaces V and W are said to be ***isomorphic***.

Chapter-8

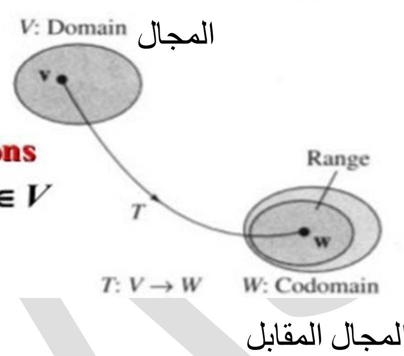
Linear Transformations

- A linear transformation is a function T that maps a vector space V into another vector space W:

$$T: V \xrightarrow{\text{mapping}} W, \qquad V, W : \text{vector space}$$

V: the domain of T

W: the co-domain of T



متى نقول أنها تحويل خطي إذا تحقق الشرطين

Two axioms of linear transformations

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$

(2) $T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$

- Image of $\mathbf{v}$ under T:

  If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that

  $T(\mathbf{v}) = \mathbf{w}$

  Then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T.

- **the range of T:**

  The set of all images of vectors in V.

  $range(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$

- the pre-image of $\mathbf{w}$:

  The set of all $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$.

- Notes:

  (1) A **linear transformation** is said to be **operation preserving**.

  $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$

  Addition in V | Addition in W | Scalar multiplication in V | Scalar multiplication in W

  (2) A linear transformation $T: V \to V$ from **a vector space into itself** is called a **linear operator**.

مثال :

$$T : R^2 \to R^2 \; v = (v_1, v_2) \in R^2 , \quad T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

a. Find the image of $v = (-1,2)$?
b. Find the pre-image of $w = (-1,11)$?

$$v = (-1,2)$$

$$\mathrm{T(v)} = \mathrm{T}(-1,2) = (-1 - 2, -1 + 2(2))$$

$$= (-3,3)$$

$$T(v) = w = (-1,11)$$

We know $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1,11)$

نحول المسألة إلى معادلتين ... لنتمكن من إيجاد القيم

$$\cancel{v_1} - v_2 = -1$$

$$-\cancel{v_1} - 2v_2 = -11$$

Multiple -1 and Add

$$-3v_2 = -12$$

$$v_2 = \frac{-12}{-3} = 4$$

بالتعويض في المعادلة الأولي :

$$v_1 - 4 = -1$$

$$v_1 = -1 + 4 \implies v_1 = 3$$

So (3,4) per-image of (-1,11)

General Linear Transformations :

In special case where $v = w$, Then linear transformation T is called linear operator on vector space V

مثال :

EXAMPLE 1 Matrix for a Linear Transformation

Let $T: P_1 \to P_2$ be the linear transformation defined by

$$T(p(x)) = xp(x)$$

Find the matrix for T with respect to the standard bases

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \quad \text{and} \quad B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

where

$$\mathbf{u}_1 = 1, \quad \mathbf{u}_2 = x; \quad \mathbf{v}_1 = 1, \quad \mathbf{v}_2 = x, \quad \mathbf{v}_3 = x^2$$

Solution From the given formula for T we obtain

$$T(\mathbf{u}_1) = T(1) = (x)(1) = x$$

$$T(\mathbf{u}_2) = T(x) = (x)(x) = x^2$$

By inspection, the coordinate vectors for $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ relative to B' are

$$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the matrix for T with respect to B and B' is

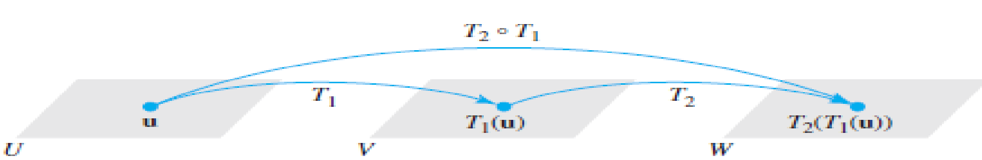
$$[T]_{B',B} = [[T(\mathbf{u}_1)]_{B'} [T(\mathbf{u}_2)]_{B'}] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Compositions Transformations :

DEFINITION 1 If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, then the *composition of T_2 with T_1*, denoted by $T_2 \circ T_1$ (which is read "T_2 circle T_1"), is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u})) \tag{1}$$

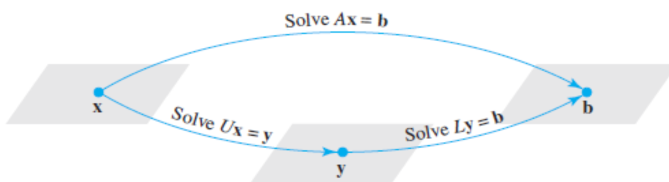
where $\mathbf{u}$ is a vector in U.



LU-Decompositions :

DEFINITION 1 A factorization of a square matrix A as $A = LU$, where L is lower triangular and U is upper triangular is called an ***LU-decomposition*** (or ***LU-factorization***) of A.

THEOREM 9.1.1 *If A is a square matrix that can be reduced to a row echelon form U by Gaussian elimination without row interchanges, then A can be factored as $A = LU$, where L is a lower triangular matrix.*



- Find a LU Decomposition :

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

القاعدة :

A = LU

$$U = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \frac{R_1}{6} \qquad\qquad L = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 9R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \qquad\qquad \begin{bmatrix} 6 & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \quad \frac{R_2}{2} \qquad\qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & x & 0 \\ 3 & x & x \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 5 \end{bmatrix} \quad R_3 \to R_3 - 8R_2 \qquad\qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & x & x \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to (1)R_3 \qquad\qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & x \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & -2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \qquad\qquad L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

مثال :

EXAMPLE 2 An *LU*-Decomposition

Find an *LU*-decomposition of

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

Step 1 $\quad \frac{1}{2} \times \text{row } 1 \qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

Step 2 $\quad (3 \times \text{row } 1) + \text{row } 2 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{bmatrix}$$

Step 3 $\quad (-4 \times \text{row } 1) + \text{row } 3 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix}$$

Step 4 $\quad (3 \times \text{row } 2) + \text{row } 3 \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$$

Step 5

$\frac{1}{7} \times$ row 3 $\quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$\begin{aligned} L &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \end{aligned}$$

so

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

is an LU-decomposition of A.

Example 1:

Let

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \end{bmatrix}$$

then, comparing the left and right hand sides row by row implies that $U_{11} = 3$, $U_{12} = 1$, $L_{21}U_{11} = -6$ which implies $L_{21} = -2$ and $L_{21}U_{12} + U_{22} = -4$ which implies that $U_{22} = -2$. Hence

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$$

is an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Example 2:

Using material from the worked example in the notes we set

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}$$

and comparing elements row by row we see that

$$\begin{array}{lll} U_{11}=3, & U_{12}=1, & U_{13}=6, \\ L_{21}=-2, & U_{22}=2, & U_{23}=-4 \\ L_{31}=0 & L_{32}=4 & U_{33}=-1 \end{array}$$

and it follows that

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

Example 3:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$.
Multiplying out LU and setting the answer equal to A gives

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

Now we have to use this to find the entries in L and U. Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that

$$\boxed{U_{11}=1}, \quad \boxed{U_{12}=2}, \quad \boxed{U_{13}=4}.$$

Now consider the second row

$$L_{21}U_{11}=3 \quad \therefore L_{21}\times 1=3 \quad \therefore \boxed{L_{21}=3},$$

$$L_{21}U_{12}+U_{22}=8 \quad \therefore 3\times 2+U_{22}=8 \quad \therefore \boxed{U_{22}=2},$$

$$L_{21}U_{13}+U_{23}=14 \quad \therefore 3\times 4+U_{23}=14 \quad \therefore \boxed{U_{23}=2}.$$

Notice how, at each step, the equation in hand has only one unknown in it, and other quantities that we have already found. This pattern continues on the last row

$$L_{31}U_{11} = 2 \quad \therefore L_{31} \times 1 = 2 \quad \therefore \boxed{L_{31} = 2},$$

$$L_{31}U_{12} + L_{32}U_{22} = 6 \quad \therefore 2 \times 2 + L_{32} \times 2 = 6 \quad \therefore \boxed{L_{32} = 1},$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13 \quad \therefore (2 \times 4) + (1 \times 2) + U_{33} = 13 \quad \therefore \boxed{U_{33} = 3}.$$

We have shown that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 4:

Let's see an example of LU-Decomposition without pivoting:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix}$$

The first step of Gaussian elimination is to subtract 2 times the first row form the second row. In order to record what was done, the multiplier, 2, into the place it was used to make a zero.

$$\xrightarrow{R2-2R1} \begin{bmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ 0 & 2 & -10 \end{bmatrix}$$

There is already a zero in the lower left corner, so we don't need to eliminate anything there. We record this fact with a (0). To eliminate a_{32}, we need to subtract -2 times the second row from the third row. Recording the -2:

$$\xrightarrow{R3-(-2)R2} \begin{bmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ (0) & (-2) & 2 \end{bmatrix}$$

Let U be the upper triangular matrix produced, and let L be the lower triangular matrix with the records and ones on the diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -10 \end{bmatrix}$$

Then,

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix} = A$$

Singular Value Decomposition

DEFINITION 1 If A is an $m \times n$ matrix, and if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A^TA, then the numbers

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \dots, \quad \sigma_n = \sqrt{\lambda_n}$$

are called the ***singular values*** of A.

THEOREM 9.5.1 *If A is an $m \times n$ matrix, then:*

(a) *A and A^TA have the same null space.*

(b) *A and A^TA have the same row space.*

(c) *A^T and A^TA have the same column space.*

(d) *A and A^TA have the same rank.*

THEOREM 9.5.2 *If A is an $m \times n$ matrix, then:*

(a) *A^TA is orthogonally diagonalizable.*

(b) *The eigenvalues of A^TA are nonnegative.*

مثال :

EXAMPLE 1 Singular Values

Find the singular values of the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution The first step is to find the eigenvalues of the matrix

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial of A^TA is

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

so the eigenvalues of A^TA are $\lambda_1 = 3$ and $\lambda_2 = 1$ and the singular values of A in order of decreasing size are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = 1$$

شرح بسيط للقيم الذاتية Eigenvalues والمتجهات الذاتيه Eigenvectors

قوانين الحل :

القانون العام $A_x = \lambda_x$

قانون لإيجاد القيمة الذاتية Eigenvalues

$$|A - \lambda I| = 0$$

أو

$$|\lambda I - A| = 0$$

قانون لإيجاد المتجهات الذاتيه Eigenvectors

$$|A - \lambda I|x = 0$$

❖ Find Eigenvalues and Eigenvectors of a 2x2 Matrix

$$A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$$

خطوات الحل لإيجاد القيمة الذاتية Eigenvalue:

1- لإيجاد دلتا مع I λ

I تعني المتطابقة Identity

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\lambda I - A| = 0$$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 5 & 3 \\ 6 & \lambda - 2 \end{bmatrix}$$

نحول المصفوفة الى المحددة وتسمى المعادلة المميزة

Characteristic Equation

لتوضيح حل المحددة :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (\mathbf{a} * \mathbf{d}) - (\mathbf{b} * \mathbf{c})$$

مع الانتباه للإشارات

$$\begin{vmatrix} \lambda - 5 & 3 \\ 6 & \lambda - 2 \end{vmatrix} = 0$$

$$= (\lambda - 5)(\lambda - 2) - (6)(3)$$

$$= \lambda^2 - 2\lambda - 5\lambda + 10 - 18$$

طريقة حل المعادلة البحث عن عددين ضربهما هو -8 ومجموعهما -7 ناتج
العددين هما +1 و -8

$$=\lambda^2 - 7\lambda - 8$$

$$\lambda^2 - 7\lambda - 8 = 0$$

المعادلة المميزة

Characteristic Equation

$$(\lambda + 1)(\lambda - 8) = 0$$

$$(\lambda + 1) = 0 \quad \Rightarrow \lambda = -1$$

$$(\lambda - 8) = 0 \Rightarrow \lambda = +8$$

The eigenvalues are $\lambda = -1$ or $\lambda = 8$

قانون لإيجاد المتجهات الذاتيه Eigenvectors

$$|\lambda I - A|x = 0$$

Put $\lambda = -1$

نأخذ القيمة $-1 = \lambda$ لإيجاد المتجهات الذاتية لهذه النقطة :

$$\begin{pmatrix} \lambda - 5 & 3 \\ 6 & \lambda - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

نعوض بدل الدلتا -1

$$\begin{pmatrix} (-1) - 5 & 3 \\ 6 & (-1) - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 3 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\begin{pmatrix} -6 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

نقل $3x_2$ الى الطرف الثاني مع تغير الاشارة

$$-6x_1 + 3x_2 = 0$$

$$-6x_1 = -3x_2$$

نعوض في أحد المتغيرات x_1 بأي رقم ماعدا 0

$$x_1 = 1$$

$$(-6)(1) = -3x_2$$

$$\frac{-6}{-3} = \frac{-3x_2}{-3}$$

بقسمة الطرفين على 3-

$$x_2 = 2$$

An eigenvector $\begin{pmatrix}1\\2\end{pmatrix}$

نأخذ القيمة الذاتية 8 = ⅄ لإيجاد المتجهات الذاتية لهذه النقطة :

Put ⅄= 8

$$|\lambda I - A|x = 0$$

نعوض بدل الدلتا 8

$$\begin{pmatrix}\lambda -5 & 3\\ 6 & \lambda -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}8-5 & 3\\ 6 & 8-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}3 & 3\\ 6 & 6\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$R_2 \to R_2 - 2R_1$$

$$\begin{pmatrix}3 & 3\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$3x_1 + 3x_2 = 0$$

نقل $3x_2$ الى الطرف الثاني مع تغير الاشارة

$$3x_1 = -3x_2$$

نعوض في أحد المتغيرات x_2 بأي رقم ماعدا 0

$$x_2 = -1$$

$$3x_1 = -3(1)$$

$$\frac{3x_1}{3} = \frac{-3}{3}$$

بقسمة الطرفين على 3

$$x_1 = 1$$

An eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$

Eigenvalues and Eigenvectors :

n × n

$Ax = \lambda x$

Characteristic Equation:

n × n

$det(\lambda I - A) = 0$

has nontrivial solution

مهم جدا

مثال على ال Eigenvalues and Eigenvectors :

EXAMPLE 2 Finding Eigenvalues

In Example 1 we observed that $\lambda = 3$ is an eigenvalue of the matrix

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

مربعه Matrix
حجمها 2 × 2

but we did not explain how we found it. Use the characteristic equation to find all eigenvalues of this matrix.

Solution It follows from Formula 1 that the eigenvalues of A are the solutions of the equation $\det(\lambda I - A) = 0$, which we can write as

اول خطوه نوجد هالقانون

لندا عباره عن حرف اغريقي وتعني رقم ثابت مثل ال scalar

حرف ال I المقصود فيه مصفوفة الوحده وتكون نفس حجم ال matrix بالسؤال

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

from which we obtain

$$(\lambda - 3)(\lambda + 1) = 0 \qquad (2)$$

اسمها characteristic polynomial

This shows that the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. Thus, in addition to the eigenvalue $\lambda = 3$ noted in Example 1, we have discovered a second eigenvalue $\lambda = -1$.

$$\begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

نعوض بال lampda ونطلع ال system

If $\lambda = 3$, then this equation becomes

$$\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

هذا Eigenvalues

whose general solution is

$$x_1 = \frac{1}{2}t, \quad x_2 = t$$

(verify) or in matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

هذا Eigenvectors

Thus,

$$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 3$. We leave it as an exercise for you to follow the pattern of these computations and show that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = -1$.

: Positive Dominant Eigenvalue, λ

THEOREM 9.2.2 *Let A be a symmetric $n \times n$ matrix with a positive dominant* eigenvalue λ. If $\mathbf{x}_0$ is a nonzero vector in R^n that is not orthogonal to the eigenspace corresponding to λ, then the sequence*

$$\mathbf{x}_0, \quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)}, \quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)}, \ldots, \quad \mathbf{x}_k = \frac{A\mathbf{x}_{k-1}}{\max(A\mathbf{x}_{k-1})}, \ldots \tag{8}$$

converges to an eigenvector corresponding to λ, and the sequence

$$\frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1}, \quad \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2}, \quad \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3}, \ldots, \quad \frac{A\mathbf{x}_k \cdot \mathbf{x}_k}{\mathbf{x}_k \cdot \mathbf{x}_k}, \ldots \tag{9}$$

converges to λ.

The Power Method:

DEFINITION 1 If the *distinct* eigenvalues of a matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_k$, and if $|\lambda_1|$ is larger than $|\lambda_2|, \ldots, |\lambda_k|$, then λ_1 is called a ***dominant eigenvalue*** of A. Any eigenvector corresponding to a dominant eigenvalue is called a ***dominant eigenvector*** of A.

normalized power squence :

$$x_0\ , x_1 = \frac{Ax_0}{\|Ax_0\|}\ , x_2 = \frac{Ax_1}{\|Ax_1\|}\ , \ldots . x_k = \frac{Ax_{k-1}}{\|Ax_{k-1}\|}$$

مثال :

EXAMPLE 2 The Power Method with Euclidean Scaling

Apply the power method with Euclidean scaling to

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{with} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Stop at $\mathbf{x}_5$ and compare the resulting approximations to the exact values of the dominant eigenvalue and eigenvector.

Solution We will leave it for you to show that the eigenvalues of A are $\lambda = 1$ and $\lambda = 5$ and that the eigenspace corresponding to the dominant eigenvalue $\lambda = 5$ is the line represented by the parametric equations $x_1 = t, x_2 = t$, which we can write in vector form as

$$\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

Setting $t = 1/\sqrt{2}$ yields the normalized dominant eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 0.707106781187\ldots \\ 0.707106781187\ldots \end{bmatrix} \tag{7}$$

Now let us see what happens when we use the power method, starting with the unit vector $\mathbf{x}_0$.

$$A\mathbf{x}_0 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \frac{1}{3.60555}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix}$$

$$A\mathbf{x}_1 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix} \approx \begin{bmatrix} 3.60555 \\ 3.32820 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \frac{1}{4.90682}\begin{bmatrix} 3.60555 \\ 3.32820 \end{bmatrix} \approx \begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix}$$

$$A\mathbf{x}_2 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix} \approx \begin{bmatrix} 3.56097 \\ 3.50445 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \frac{1}{4.99616}\begin{bmatrix} 3.56097 \\ 3.50445 \end{bmatrix} \approx \begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix}$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix} \approx \begin{bmatrix} 3.54108 \\ 3.52976 \end{bmatrix} \qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \frac{1}{4.99985}\begin{bmatrix} 3.54108 \\ 3.52976 \end{bmatrix} \approx \begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix}$$

$$A\mathbf{x}_4 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix} \approx \begin{bmatrix} 3.53666 \\ 3.53440 \end{bmatrix} \qquad \mathbf{x}_5 = \frac{A\mathbf{x}_4}{\|A\mathbf{x}_4\|} \approx \frac{1}{4.99999}\begin{bmatrix} 3.53666 \\ 3.53440 \end{bmatrix} \approx \begin{bmatrix} 0.70733 \\ 0.70688 \end{bmatrix}$$

$$\lambda^{(1)} = \left(A\mathbf{x}_1\right) \cdot \mathbf{x}_1 = (A\mathbf{x}_1)^T\mathbf{x}_1 \approx [3.60555 \quad 3.32820]\begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix} \approx 4.84615$$

$$\lambda^{(2)} = \left(A\mathbf{x}_2\right) \cdot \mathbf{x}_2 = (A\mathbf{x}_2)^T\mathbf{x}_2 \approx [3.56097 \quad 3.50445]\begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix} \approx 4.99361$$

$$\lambda^{(3)} = \left(A\mathbf{x}_3\right) \cdot \mathbf{x}_3 = (A\mathbf{x}_3)^T\mathbf{x}_3 \approx [3.54108 \quad 3.52976]\begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix} \approx 4.99974$$

$$\lambda^{(4)} = \left(A\mathbf{x}_4\right) \cdot \mathbf{x}_4 = (A\mathbf{x}_4)^T\mathbf{x}_4 \approx [3.53666 \quad 3.53440]\begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix} \approx 4.99999$$

$$\lambda^{(5)} = \left(A\mathbf{x}_5\right) \cdot \mathbf{x}_5 = (A\mathbf{x}_5)^T\mathbf{x}_5 \approx [3.53576 \quad 3.53531]\begin{bmatrix} 0.70733 \\ 0.70688 \end{bmatrix} \approx 5.00000$$

Thus, $\lambda^{(5)}$ approximates the dominant eigenvalue to five decimal place accuracy and $\mathbf{x}_5$ approximates the dominant eigenvector in 7 correctly to three decimal place accuracy.

Diagonalization:

$$B = P^{-1}AP$$

ملاحظه :

ال diagonalizable matrix P is side to diagonalize A

ال property:

Same determinant - Same Invertibility - Same Rank - Same Nullity - Same Trace - Same characteristic polynomial - Same Eigenvalues - Same Eigenspace dimension.

مثال على Diagonalization :

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution In Example 7 of the preceding section we found the characteristic equation of A

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and we found the following bases for the eigenspaces:

$$\lambda = 2: \ \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 1: \ \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. As a check, you should verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 21 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Complex Vector Spaces :

ال complex number معناها رقم تخيلي ورقم حقيقي

$$z = a + bi$$

عباره عن عدد تخيلي

$$\bar{z} = a - bi$$

Called complex conjugate

$$z\bar{z} = a^2 - b^2 = |z|^2$$

Properties of the Complex Conjugate :

DEFINITION 1 If A is a complex matrix, then the ***conjugate transpose*** of A, denoted by A^*, is defined by

$$A^* = \overline{A}^T \tag{1}$$

THEOREM 5.3.1 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* C^n, *and if* k *is a scalar, then:*

(a) $\overline{\overline{\mathbf{u}}} = \mathbf{u}$

(b) $\overline{k\mathbf{u}} = \overline{k}\overline{\mathbf{u}}$

(c) $\overline{\mathbf{u}+\mathbf{v}} = \overline{\mathbf{u}} + \overline{\mathbf{v}}$

(d) $\overline{\mathbf{u}-\mathbf{v}} = \overline{\mathbf{u}} - \overline{\mathbf{v}}$

THEOREM 5.3.2 *If* A *is an* $m \times k$ *complex matrix and* B *is a* $k \times n$ *complex matrix, then:*

(a) $\overline{\overline{A}} = A$

(b) $(\overline{A^T}) = (\overline{A})^T$

(c) $\overline{AB} = \overline{A}\,\overline{B}$

THEOREM 7.5.1 *If* k *is a complex scalar, and if* A, B, *and* C *are complex matrices whose sizes are such that the stated operations can be performed, then:*

(a) $(A^*)^* = A$

(b) $(A+B)^* = A^* + B^*$

(c) $(A-B)^* = A^* - B^*$

(d) $(kA)^* = \overline{k}A^*$

(e) $(AB)^* = B^*A^*$

مثال :

Find the complex conjugate :

$5+3i$

Answer:

$5-3i$ ← فقط نغير إشارة الحد التخيلى

مثال على complex number:

مثال على الجمع :

$$(3-5i)+(6-7i)$$

$$(3-5i)+(6-7i)= (3+6)+(-5+7)i=9+2i$$

مثال على الطرح :

$$(3-5i)-(6-7i)$$

$$(3-5i)-(6-7i)= (3-6)+(-5-7)i=-3-12i$$

مثال على الضرب :

$$(2-i)(3+4i)$$

$$(2-i)(3+4i)=(2)(3)+(2)(4i)+(-i)(3)+(-i)(4i)$$

$$=6+8i-3i-4i^2=6+5i-4(-1)$$

$$=6+5i+4=10+5i$$

$$\frac{3}{2i}=\frac{3}{2i}\cdot\frac{i}{i}=\frac{3i}{2i^2}=\frac{3i}{2(-1)}=\frac{3i}{-2}=\frac{3i}{2}=\frac{3}{2}i$$

$$\frac{1}{4+2i}=\frac{1}{4+2i}\cdot\frac{4-2i}{4-2i}=\frac{4-2i}{16+4}=\frac{4}{20}-\frac{2}{20}i=\frac{2}{10}-\frac{1}{10}i$$

مثال :

EXAMPLE 1 Conjugate Transpose

Find the conjugate transpose A^* of the matrix

$$A=\begin{bmatrix}1+i & -i & 0\\ 2 & 3-2i & i\end{bmatrix}$$

Solution We have

$$\overline{A}=\begin{bmatrix}1-i & i & 0\\ 2 & 3+2i & -i\end{bmatrix} \text{ and hence } A^*=\overline{A}^T=\begin{bmatrix}1-i & 2\\ i & 3+2i\\ 0 & -i\end{bmatrix}$$

Hermitian Matrices :

DEFINITION 2 A square complex matrix A is said to be *unitary* if

$$A^{-1} = A^* \tag{3}$$

and is said to be *Hermitian** if

$$A^* = A \tag{4}$$

THEOREM 7.5.2 *The eigenvalues of a Hermitian matrix are real numbers.*

مثال :

EXAMPLE 2 Recognizing Hermitian Matrices

Hermitian matrices are easy to recognize because their diagonal entries are real (why?), and the entries that are symmetrically positioned across the main diagonal are complex conjugates. Thus, for example, we can tell by inspection that

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{bmatrix}$$

لازم يكون قطرها real numbers
وحولها الأرقام متناظره

is Hermitian

Quadratic Forms :

DEFINITION 1 A quadratic form $\mathbf{x}^T A\mathbf{x}$ is said to be

positive definite if $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$

negative definite if $\mathbf{x}^T A\mathbf{x} < 0$ for $\mathbf{x} \neq \mathbf{0}$

indefinite if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values

مثال :

السؤال عليها أوجد ال matrix من هذي ال System

هذي معادله 2 × 2

$$2x^2+6xy-5y^2=\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x_1^2+7x_2^2-3x_3^2+4x_1x_2-2x_1x_3+8x_2x_3=[x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

هذي معادله 3 × 3

اذا كانت ال system 3 × 3 ف ال main diagonal
او قطرها هو معاملات التربيع

وباقي المعاملات نقسمهم على 2 ونوزعهم حول القطر
بشكل متناظر

مثال :

Solving Quadratic equations by factoring :

4 أمثله

1

$$x^2+9x+14=(x+7)(x+2)$$

$$7 \bullet 2 = 14$$

$$7+2=9$$

2

$$x^2-9x+14=(x-7)(x-2)$$

$$(-7) \bullet (-2) = 14$$

$$(-7)+(-2)=-9$$

$\boxed{3}$

$$x^2 + 5x - 14 = (x + 7)(x - 2)$$

$$7 \bullet (-2) = -14$$

$$7 + (-2) = +5$$

$\boxed{4}$

$$x^2 - 5x - 14 = (x - 7)(x + 2)$$

$$(-7) \bullet 2 = -14$$

$$(-7) + 2 = -5$$

Quadratic Forms : نماذج من الدرجة الثانية (التربيعي)

Express the quadratic forms in matrix , then the associated symmetric

$$3x_1^2 + 2x_2^2 - 4x_3^2 - 2x_1x_2 + 6x_1x_3 - 4x_3x_1$$

الحل :

$$\begin{bmatrix} 3 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & -4 \end{bmatrix}$$

المرفوع إلى القوة تضعها في القطر حسب الرقم المحدد بالوان الصفر يعني الصفوف

- يعني الصف الاول والعمود الاول
- يعني الصف الثاني والعمود الثاني
- يعني الصف الثالث والعمود الثالث

بالقسمة على $\frac{1}{2}$ لكل رقم

- يعني الصف الاول والعمود الثاني
- يعني الصف الاول والعمود الثالث
- يعني الصف الثالث والعمود الاول

Conjugate transpose

If A is a complex matrix, then the ***conjugate transpose*** of A, denoted by A^* , is defined by

$$A^* = \bar{A}^{\,T}$$

Find the conjugate transpose A^* of the matrix

$$A = \begin{bmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{bmatrix}$$

الحل :

حسب القانون $A^* = \bar{A}^{\,T}$

1- نجد نفي $\bar{A}$ ننظر إلى المتغير في المصفوفة نغيير إشارة

$$\bar{A} = \begin{bmatrix} 1-i & i & 0 \\ 2 & 3+2i & -i \end{bmatrix}$$

2- نجد تبديل المصفوفة المنفية $\bar{A}^{\,T}$ *يعني نحول الصف الاول في العمود الاول وكذلك الصف الثاني في العمود الثاني*

$$\bar{A}^{\,T} = \begin{bmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{bmatrix}$$

Unitary Matrices :

Definition of a Unitary Matrix

A complex matrix A is **unitary** if

$$A^{-1} = A^*.$$

$$AA^* = A^*A = I$$

Show that the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ **is unitary.**

$AA^* = I$

$A^* = \overline{A}^T$

$$\overline{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\overline{A}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$AA^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(-\frac{1-i}{\sqrt{3}}\right)\left(-\frac{1+i}{\sqrt{3}}\right) & \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1-i}{\sqrt{3}}\right) + \left(-\frac{1+i}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) \\ \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1+i}{\sqrt{3}}\right) & \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{1-i}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) \end{bmatrix}$$

Find the maximum values of $P = 3x + 2y$ subject to

$$x + 4y \leq 20$$

$$2x + 3y \leq 30$$

$$x \geq 0, y \geq 0$$

Solution:

1. Graph the feasible region.

A- Start with the line $x + 4y = 20$.

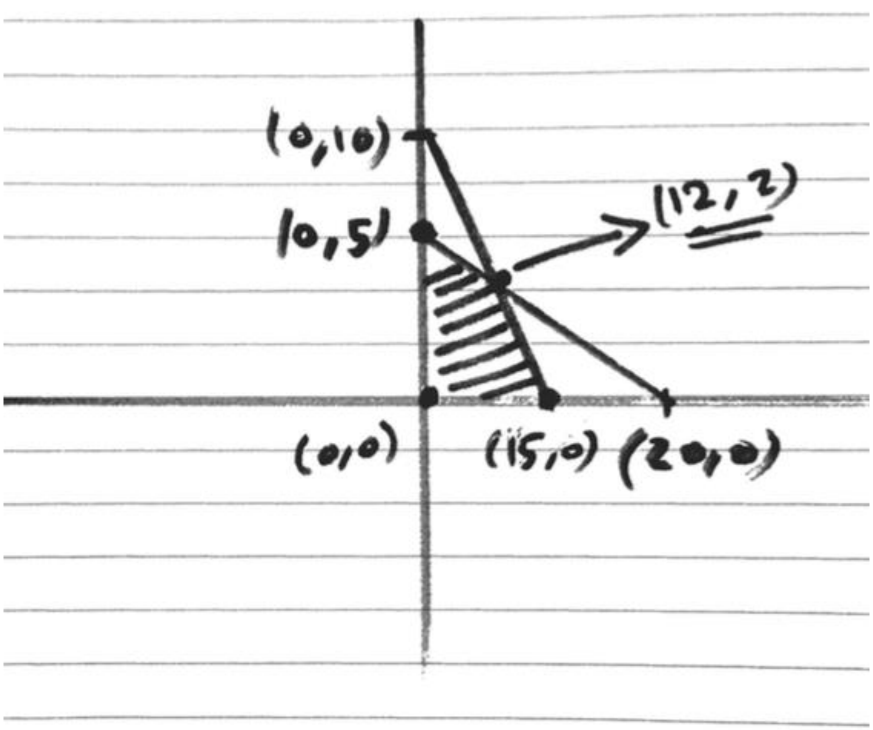
i- Find the intercepts by letting $x = 0$ and $y = 0$: $(0, 5)$ and $(20, 0)$.

ii- Test the origin: $(0) + 4(0) \leq 20$. This is true, so we keep the half-plane containing the origin.

B- Now the line $2x + 3y = 30$.

i- Find the intercepts $(0, 10)$ and $(15, 0)$.

ii- Test the origin $2(0) + 3(0) \leq 30$ it is true.



2. Find the corner points.

We find (0,0) is one corner.

Also, (0,5) is the corner from the y-intercept of the first equation.

The corner (15,0) is from the second equation.

We can find the fourth corner which is the intersection of the two lines using intersect on the calculator or solving the two equations:

$$x + 4y = 20$$

$$2x + 3y = 30$$

by elimination, we get $(x, y) = (12, 2)$.

3. Evaluate the objective function at each vertex.

Put the vertices into a table:

Point	$P = 3x + 2y$
(0, 0)	0 Min
(0, 5)	10
(15, 0)	45 Max
(12, 2)	40

4. The region is bounded; therefore a max and a min exist. The minimum is at the point (0,0) with a value of P=0 . The maximum is at the point (15,0) and the value is P=45.

2. Select one of the alternatives from the following questions as your answer.

(a) The characteristic equation of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ is

A. $\lambda^2 - 7\lambda - 10 = 0$
B. $\lambda^2 + 7\lambda - 10 = 0$
C. $\lambda^2 - 7\lambda + 10 = 0$
D. $\lambda^2 + 7\lambda + 10 = 0$

معنى السؤال ماهي المعادلة المميزة في المصفوفة : حل الاختيار
من قانون eigenvalue نجد المعادلة ..

$$|\lambda I - A| = 0$$

نوجد المحددة

$$\begin{vmatrix} \lambda - 3 & -1 \\ -2 & \lambda - 4 \end{vmatrix} = 0$$

اضرب الاقواس

$$(\lambda - 3)(\lambda - 4) - (-1)(-2) = 0$$

اجمع

$$\lambda^2 - 4\lambda - 3\lambda + 12 - 2 = 0$$

$$\boxed{\lambda^2 - 7\lambda + 10 = 0}$$

نلاحظ أن المصفوفة A مرفوعة الى قوة 3
يعنى أن $(1)^3, (4)^3, (-3)^3$

(b) The eigenvalues of the matrix A^3, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & -3 \end{bmatrix}$, are

A. $\{1, 4, -3\}$
B. $\{1, 12, -9\}$
C. $\{1, 64, 27\}$
D. $\{1, 64, -27\}$

لإيجاد القيمة الذاتية Eigenvalueفي المثلث الصفري
العلوي أو السفلي نأخذ القطر المحدد بالوان الاحمر.

(c) Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$:

A. (1,2), (-2,1)
B. (3,4),(2,6)
C. (6,9),(5,2)
D. (0,4), (0,6)

قيمة المتجه المتعامد تساوي صفر

$$u \cdot v = 0$$
$$u \cdot v = (1)(-2) + (2)(1)$$
$$= -2 + 2 = 0$$

شرح الواجب الرابع :

السؤال الثاني:

2. Select one of the alternatives from the following questions as your answer.

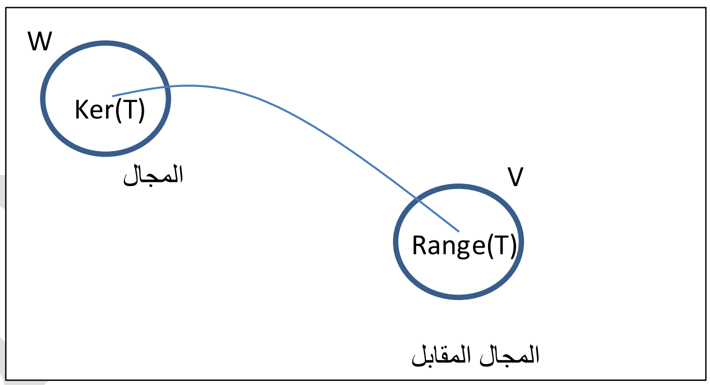
(a) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given by $T(x, y) = (2x - y, -4x + 2y)$, then which of the following vector is in $ker(T)$?

A. $(1, 4)$
B. $(2, 1)$
C. $(1, 1/2)$
D. $(1/2, 1)$

نوجد ker(T)=0
المعادلتين متساويتين
نأخذ أحد المعادلتين ونعوض فيها بأحد
النقاط التي في الخيارات

(b) If $T : W \to V$ be a linear transformation, then $ker(T)$ and $range(T)$ are subspaces of vector space(s)

A. V.
B. W.
C. W and V respectively.
D. V and W respectively.



(c) Which of the following sets of eigenvalues have a dominant eigenvalue:

A. $\{8, -7, -6, 8\}$
B. $\{-5, -2, 2, 4\}$
C. $\{-3, -2, -1, , 0, 1, 2, 3\}$
D. None of the above

نأخذ اكبر قيمة مطلقة في المجموعة ثم نقارنها بالقيم الأخرى بشرط أن القيمة المطلقة غير متكررة في القيم الذاتية

الا ن **B** اكبر قيمة مطلقة هي -5 وغير مكرره

(d) If $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ be a matrix where $B = A^T A$, then the singular values of A are

A. $\{4, 9, 0\}$
B. $\{0, 9, 16\}$
C. $\{4,9,16\}$
D. $\{2, 3, 4\}$

(e) In linear programming, objective function and objective constraints are

A. solved.
B. quadratic.
C. adjacent.
D. linear.

(f) The feasible region

A. is defined by the objective function.
B. is an area bounded by the collective constraints and represents all permissible combinations of the decision variables.
C. represents all values of each constraint.
D. may range over all positive or negative values of only one decision variable.

هي المنطقة المحصورة بين النقاط المحددة وتمثل كل المجموعات المسموح بها من متغيرات.

Consider the basis $S = \{v_1, v_2\}\ for\ \mathbb{R}^2$, where $v_1 = \{1,1\}$ and $v_2 = \{1,0\}$ and let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear operator for which $T(v_1) = (1,2)\ and\ T(v_1) = (3,0)$.

Find a formula for $T(x_1, x_2)$ and use it to find $T(2, -4)$.

Solution :

- Find a formula for $T(x_1, x_2)$?

$$(x_1, x_2) = c_1 v_1 + c_2 v_2$$

$$= c_1(1,1) + c_2(1,0)$$

$$= (c_1, c_1) + (c_2, 0)$$

$$c_1 + c_2 = x_1 \quad \Rightarrow \quad x_2 + c_2 = x_1 \quad \Rightarrow \quad c_2 = x_1 - x_2$$

$$c_1 = x_2$$

$$(x_1, x_2) = x_2(1,1) + x_1 - x_2(1,0)$$

$$T(x_1, x_2) = x_2 T(1,1) + (x_1 - x_2) T(1,0)$$

$$= (x_2)(1,2) + (x_1 - x_2)(3,0)$$

$$= (x_2, 2x_2) + (3x_1 - 3x_2, 0)$$

$$= (x_2 + 3x_1 - 3x_2\ ,\ \ 2x_2 + 0)$$

$$T(x_1, x_2) = (3x_1 - 2x_2\ , 2x_2)$$

بالتعويض في المعادلة لإيجاد النقطة

- find $T(2, -4)$?

$$T(2, -4) = ((3)(2) - (2)(-4)\ \ ,\ \ 2(-4))$$

=(12 , -8)

. Find the singular values of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Solution:

$$A^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$|\lambda I - A^T A| = 0$$

$$\begin{vmatrix} \lambda - 5 & -4 \\ -4 & \lambda - 5 \end{vmatrix} = 0$$

$$(\lambda - 5)(\lambda - 5) - (-4)(-4) = 0$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$(\lambda - 1)(\lambda - 9) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 9$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{1} = 1$$

Are not the Signaler values.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3$$

By : Atheel **Exercises Chapter (6) – week (10)**

Find least squares solution ?

System المعطى

$$x_1 - x_2 = 4$$
$$3x_1 + 2x_2 = 1$$
$$-2x_1 + 4x_2 = 3$$

step ① :- b و A نطلع system من

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

step ②:- $\boxed{A^T \cdot Ax = A^T \cdot b}$ نطبق القانون حفظ

نوجدها ، في السؤال نوجدها في السؤال

$$A^T = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}$$ * قلب الصفوف أعمدة *

$\boxed{A^T \cdot Ax = A^T \cdot b}$ * نطبق القانون بالضبط كما هو

$$\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

مضروبه حسب القانون ب $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

ضرب عادي ضرب عادي

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

نختارها (2x1) حتى تكون عمليه لضـ متوازن

$$(2\times2) \cdot (2\times1) = (2\times1)$$

* الان نوجد المعادلات بقانون التعويض

$$14x_1 - 3x_2 = 1 \Rightarrow 7 \text{ نضرب في}$$
$$-3x_1 + 21x_2 = 10$$

$$98x_1 - \cancel{21x_2} = 7$$
$$-3x_1 + \cancel{21x_2} = 10$$

* الان نوجد x_2 نعوض في احدى المعادلات

* من المعادلة الاولى $14\left(\frac{17}{95}\right) - 3x_2 = 1$

$$\frac{238}{95} - 3x_2 = 1$$

* الان حصلنا على $\boxed{\frac{17}{95} = x_1}$ و $\boxed{\frac{143}{285} = x_2}$

انتهى الحل

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* Find error vector ??

* ايجادها من قانون ثابت حفظ

$$\boxed{\| b - A x \|}$$

اوجدناها من السؤال

من السؤال اوجدناها

اوجدناها x_1 و x_2

* بالتطبيق مباشرة من القانون

$$= \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} =$$

نضرب اولا كما اخذنا سابقا

$$= \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{-92}{285} \\ \frac{439}{285} \\ \frac{95}{57} \end{bmatrix} = \begin{bmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{bmatrix}$$

* هذا الجواب

عملية طرح عادية

كل عدد مع المقابل له

*Find error ??

$$\boxed{\| b - Ax \|}$$

* ايجاد error من قانون ثابت حفظ

* من الخطوة السابقة في ايجاد error vector نشيش بها ونوجد norm ⇐ ‖ ‖

⇐ أخذناها سابقاً

$$\| b - Ax \| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(\frac{-154}{285}\right)^2 + \left(\frac{4}{3}\right)^2} \simeq 4.556$$

هذه الاعداد من الخطوة السابقة في ايجاد error vector ناتجها

الناتج النهائي

Q- Find Eigenvalues ?

من اسمها الناتج قيم

المعطى → $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ → لازم تكون $n \times n$

step 1 → $\boxed{\det(\lambda I - A) = 0}$ ← نطبق هذا القانون

اذاً
A → موجودة في السؤال
I → $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ معروفة سابقاً لابد نفس مقاس A $\underline{2 \times 2}$ I
λ → ممكن تسميها اي اسم أهم شئ انها عدد ثابت اما a, b, ...

☆ $\boxed{\det(\lambda I - A) = 0}$ نطبق القانون ☆

$$\det\left(\underset{\substack{\downarrow}}{\lambda} \cdot \underset{\substack{\downarrow \\ I}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} - \underset{\substack{\downarrow \\ A}}{\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}} \right) = 0$$

عدد ثابت يضرب في ماتركس I

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \right) = 0$$

ضربنا λ في I

$$\det\left(\begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix} \right) = 0 \Rightarrow$$

طرحنا ناتج $(A) - (\lambda \cdot I)$
الان نوجد determenate

⇐ بما ان الماتركس 2x2 اذاً علينا ايجاد det عن طريق ضرب الأقطار, طرح ضرب الاخر بعض

$$\det = (\lambda-3)(\lambda+1) - (0)(-8) = 0$$
$$\det = (\lambda-3)(\lambda+1) = 0$$

من هذا الناتج نوجد قيم λ
وهي نفسها قيم $\underline{\text{Eigenvalues}}$

$$\left.\begin{aligned} \lambda - 3 = 0 &\Rightarrow \boxed{\lambda = 3} \\ \lambda + 1 = 0 &\Rightarrow \boxed{\lambda = -1} \end{aligned}\right\}$$

☆ لدينا قيمتان لـ λ Eigenvalue

1

Q - Find eigen vectors ?

* من السؤال السابق لدينا $\boxed{\lambda = 3}$ و $\boxed{\lambda = -1}$ اذاً نوجد eigen Vectors

① نوجد system

$$\begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⇐ مثل القانون ثابت

المعادلة قبل إيجاد det في السؤال السابق

نحن نكتبها لابد تكون 2x1 لأنها ستضرب في 2x2

2x1 لابد تساوي الزيرو ماتركس حتى نوجد قيم eigenvectors

$$(2 \times \cancel{2}) \cdot (\cancel{2} \times 1) = 2 \times 1$$

شكل الماتركس حقة القانون لابد نختار ال size لهذا الشكل حتى يتم الضرب بطريقة صحيحة ..

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

② نعوض في المعادلة عن $\boxed{3 = \lambda}$

$$= \begin{bmatrix} 3-3 & 0 \\ -8 & 3+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{0} & \boxed{0} \\ \boxed{-8} & \boxed{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

③ نوجد المعادلات من ↗

* $0x_1 + 0x_2 = 0$ * ← المعادلة الاولى

* $-8x_1 + 4x_2 = 0$ * ← المعادلة الثانية

طريقة إيجاد المعادلة: ضرب كل عنصر من ① ب x_1 ثم ② ب x_2

* هنا المعادلة الاولى كلها أصفار ، نحذفها ونأخذ المعادلة الثانية

$$-8x_1 + 4x_2 = 0$$

$$-8x_1 = -4x_2 \Rightarrow$$ قسمنا على 8-

$$\boxed{x_1 = \frac{1}{2} x_2}$$

②

$\boxed{x_1 = \frac{1}{2} x_2}$

$\boxed{x_1 = \frac{1}{2} t}$ $\qquad$ $\boxed{t = \frac{1}{2} x_2}$

* الان نعوض كن $\boxed{t = 1}$ او اي عدد

$\boxed{x_1 = \frac{1}{2}(1)}$ $\qquad$ $\boxed{1 = \frac{1}{2} x_2}$ دون اعتبار للثابت نأخذ الطرف ككل

$\Downarrow$

$\boxed{x_1 = \frac{1}{2}}$ $\qquad$ $\boxed{1 = x_2}$

الآن النتيجتان $\boxed{\frac{1}{2} = x_1}$ و $\boxed{x_2 = 1}$ في matrix حتى نوجد

eigen vector $\Rightarrow$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ $\rightarrow$ هذه اول eigenvector *

لو عوضنا عن عدد آخر لـ t سوف نوجد حل آخر $\boxed{t = 2}$

$\boxed{t = 2} \Rightarrow$

$\rightarrow x_1 = \frac{1}{2}(t) \Rightarrow \boxed{x_1 = \frac{1}{2}(2)} \Rightarrow \boxed{x_1 = 1}$

$\rightarrow t = \frac{1}{2} x_2 \Rightarrow \boxed{2 = \frac{1}{2} x_2} \Rightarrow \boxed{x_2 = 2}$

دون اعتبار للثابت

هذا حل آخر لـ eigenvector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

* ونكرر نفس الخطوات السابقة لـ $\boxed{\lambda = -1}$ ...

Q Find a Matrix $\underline{P}$ that digonalizes $\underline{A}$?

* اولاً ايجاد eigen value و eigenvector
* اوجدناها في السؤال السابق

من

$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Leftarrow \boxed{\lambda = 3}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftarrow \boxed{\lambda = -1}$

* لإيجاد P ⇐ اولاً .. اجمع eigen vector في matrix وحدة فقط

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

* لإيجاد B من القانون $\boxed{B = P^{-1} A P}$

نوجدها الآن ↙ في السؤال اوجدناها سابقاً

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}$$

أخذناها سابقاً

$$P^{-1} = \frac{1}{(0 \times 1) - (\frac{1}{2} \times 1)} \begin{bmatrix} 1 & \frac{-1}{2} \\ -1 & 0 \end{bmatrix} = \frac{1}{-\frac{1}{2}} \begin{bmatrix} 1 & \frac{-1}{2} \\ -1 & 0 \end{bmatrix} = -2 \begin{bmatrix} 1 & \frac{-1}{2} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \quad \boxed{B = P^{-1} A P}$$

بتطبيق مباشرة

باختصار بدون ان تتم عملية الضرب المطولة الناتج مباشرة عبارة عن digonal Matrix بمعنى القطر أرقام والبقية اصفار

$$B = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

الارقام هي عبارة عن

$3 = \lambda$

$-1 = \lambda$

Hessah Aldayel- SEU-MATH251

Chapter 1 (systems of liner)

True-False Exercises

(1) A **linear system whose equations are all homogeneous must be consistent. True**

(2) Multiplying a linear equation through by zero is an acceptable elementary row operation. False

(3) The linear system

$$x - y = 3$$

$$2x - 2y = k$$

cannot have a unique solution, regardless of the value of k. True

(4) A **single linear equation with two or more unknowns must always have infinitely many solutions. True**

(5) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent. False

(6) If each equation in a consistent linear system is multiplied through by a constant c**, then all solutions to the new system can be obtained by multiplying solutions from the original system by** c**. False**

(7) Elementary row operations permit one equation in a linear system to be subtracted from another. True

(8) The linear system with corresponding augmented matrix is consistent. False

(9) If a matrix is in reduced row echelon form, then it is also in row echelon form. True

(10) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form. False

(11) Every matrix has a unique row echelon form. False

(12) A **homogeneous linear system in** n **unknowns whose corresponding augmented matrix has a reduced row echelon form with** r **leading 1's has** $n - r$ **free variables. True**

(13) All leading 1's in a matrix in row echelon form must occur in different columns. True

(14) If every column of a matrix in row echelon form has a leading 1 then all entries that are not leading 1's are zero. False

(15) If a homogeneous linear system of n **equations in** n **unknowns has a corresponding augmented matrix with a reduced row echelon form containing** n **leading 1's, then the linear system has only the trivial solution. True**

(16) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions. False

(17) If a linear system has more unknowns than equations, then it must have infinitely many solutions. False

(18) The matrix $\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}$ has no main diagonal. True

(19) An $m \times n$ matrix has m column vectors and n row vectors. False

(20) If A and B $2X2$ are matrices, then $AB = BA$. False

(21) The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B. False

(22) For every matrix A, it is true that $\left(A^T\right)^T = A$. True

(23) If A and B are square matrices of the same order, then $tr(AB) = tr(A)tr(B)$. False

(24) If A and B are square matrices of the same order, then $(AB)^T = A^T B^T$. False

(25) For every square matrix A, it is true that $tr(A^T) = tr(A)$. True

(26) If A is a $6X4$ matrix and B is an $m \times n$ matrix such that $B^T A^T$ is $2X6$ a matrix, then $m = 4$ and $n = 2$

True

(27) If A is an $n \times n$ matrix and c is a scalar, then $tr(cA) = c\,tr\,(A)$. True

(28) If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$. True

(29) If A, B, and C are square matrices of the same order such that $AC = BC$, then $A = B$. False

(30) If $AB + BA$ is defined, then A and B are square matrices of the same size. True

(31) If B has a column of zeros, then so does AB if this product is defined. True

(32) If B has a column of zeros, then so does BA if this product is defined. False

(33) Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$ False

(34) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$. False

(35) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$. False

(36) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$

False

(37) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$. False

(38) The matrix

$$A = \begin{matrix} a & b \\ c & d \end{matrix}$$

is invertible if and only if $ad - bc \neq 0$. True

(39) If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$. True

(40) If A is an invertible matrix, then so is A^T .True

(41) If $p(x) = a_0 + a_1x + a_2x^2 + \cdots. + a_m\, x^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots..a_m$ False

(42) A square matrix containing a row or column of zeros cannot be invertible. True

(43) The sum of two invertible matrices of the same size must be invertible. False

(44) The product of two elementary matrices of the same size must be an elementary matrix. False

(45) Every elementary matrix is invertible. True

(46) If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent. True

(47) If A is an $n\ x\ n$ matrix that is not invertible, then the linear system $Ax = 0$ has infinitely many solutions. True

(48) If A is an $n\ x\ n$ matrix that is not invertible, then the matrix obtained by interchanging two rows of A cannot be invertible. True

(49) If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible. True

(50) An expression of the invertible matrix A as a product of elementary matrices is unique. False

(51) It is impossible for a linear system of linear equations to have exactly two solutions. True

(52) If the linear system has a unique solution, then the linear system also must have a unique solution. True

(53) If A and B are $n\ x\ n$ matrices such that $AB = I$, then $BA = I\ _n$True

(54) If A and B are row equivalent matrices, then the linear systems $Ax = 0$ and $Bx = 0$ have the same solution set. True

(55) If A is an $n\ x\ n$ matrix and S is an $n\ x\ n$ invertible matrix, then if x is a solution to the linear system $(S^{-1}AS)x = b$, then Sx is a solution to the linear system $Ay = Sb$. True

(56) Let A be an $n\ x\ n$ matrix. The linear system $Ax = 4x$ has a unique solution if and only if

$A - 4\ _1$is an invertible matrix. True

(57) Let A and B be $n\ x\ n$ matrices. If A or B (or both) are not invertible, then neither is AB. True

(58) The transpose of a diagonal matrix is a diagonal matrix. True

(59) The transpose of an upper triangular matrix is an upper triangular matrix. False

(60) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix. False

(61) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal. True

(62) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal. True

(63) The inverse of an invertible lower triangular matrix is an upper triangular matrix. False

(64) A diagonal matrix is invertible if and only if all of its diagonal entries are positive. False

(65) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix. True

(66) A matrix that is both symmetric and upper triangular must be a diagonal matrix. True

(67) If A and B are $n \ x \ n$ matrices such that $A + B$ is symmetric, then A and B are symmetric. False

(68) If A and B are n x n matrices such that $A + B$ is upper triangular, then A and B are upper triangular. False

(69) If A^2 is a symmetric matrix, then A is a symmetric matrix. False

(70) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix. True

(71) The determinant of the $2X2$ $\begin{matrix} a & b \\ c & d \end{matrix}$ matrix is $ad + bc$ False

(72) Two square matrices A and B can have the same determinant only if they are the same size. False

(73) The minor M_{ij} is the same as the cofactor C_{ij} if and only if $i + j$ is even. True

(74) If A is a $3X3$ symmetric matrix, then $C_{ij} = C_{ij}$ for all i and j. True

(75) The value of a cofactor expansion of a matrix A is independent of the row or column chosen for the expansion. True

(76) The determinant of a lower triangular matrix is the sum of the entries along its main diagonal. False

(77) For every square matrix A and every scalar c, we have $det(cA) = c \, det(A)$. False

(78) For all square matrices A and B, we have $det(A + B) = c \ det \ (B)$. False

(79) For every $2X2$ matrix A, we have $det(A^2) = (det(A))^2$. True

(80) If A is a $4 \ x \ 4$ matrix and B is obtained from A by interchanging the first two rows and then interchanging the last two rows, then $det(B) = det(A)$. True

(81) If A is a $3x3$ matrix and B is obtained from A by multiplying the first column by 4 and multiplying the third column by $\frac{3}{4}$, then $det(B) = 3det(A)$. True

(82) If A is a $3x3$ matrix and B is obtained from A by adding 5 times the first row to each of the second and third rows, then $det(B) = 25det(A)$. False

(83) If A is an $n\ x\ x$ matrix and B is obtained from A by multiplying each row of A by its row number, then $det(B) = \frac{n(n+1)}{2} det\ (A)$ False

(84) If A is a square matrix with two identical columns, then $det(A) = 0$. True

(85) If the sum of the second and fourth row vectors of a $6x6$ matrix A is equal to the last row vector, then $det(A) = 0$. True

(86) If A is a $3x3$ matrix, then $det\ (2A) = 2\ det\ (A)$ .False

(87) If A and B are square matrices of the same size such that $det(A) = det(B)$, then

$det\ (A + B) = 2det(A)$ .False

(88) If A and B are square matrices of the same size and A is invertible, then $det(A^{-1}BA\ ) = det(B)$ True

(89) A square matrix A is invertible if and only if $det(A) = 0$. False

(90) The matrix of cofactors of A is precisely $[adj(A)]^{\ T}$ True

(91) For every $n\ x\ n$ matrix A, we have $A\ .\ adj(A) = (det\ (A))I_n$ True

(92) If A is a square matrix and the linear system $Ax = 0$ has multiple solutions for x, Then

$det(A) = 0$ .True

(93) If A is an $n\ x\ n$ matrix and there exists an $n\ x\ 1$ matrix b such that the linear system $Ax = b$ has no solutions, then the reduced row echelon form of A cannot be I_n. True

(94) If E is an elementary matrix, then $Ex = 0$ has only the trivial solution. True

(95) If A is an invertible matrix, then the linear system $Ax = 0$ has only the trivial solution if and only if the linear system $A^{-1}x = 0$ has only the trivial solution. True

(96) If A is invertible, then $adj(A)$ must also be invertible. True

(97) If A has a row of zeros, then so does $adj(A)$. False

(98) Two equivalent vectors must have the same initial point. False

(99) The vectors $(a\ , b)$ and $(a, b, 0)$ are equivalent. False

(100) If k is a scalar and v is a vector, then v and kv are parallel if and only if $k \geq\ 0$. False

(101) The vectors $v + (u + w)$ and $(w + v) + u$ are the same. True

(102) If $u + v\ =\ u + w$, then $v = w$. True

(103) If a and b are scalars such that $au + bv = 0$, then u and v are parallel vectors. **False**

(104) Collinear vectors with the same length are equal. **False**

(105) If $(a, b, c) + (x, y, z) = (x, y, z)$, then (a, b, c) must be the zero vector. **True**

(106) If k and m are scalars and u and v are vectors, then $(k + m)(u + v) = ku + mv$ **False**

(107) If the vectors v and w are given, then the vector equation $3(2v - x) = 5x - 4w + v$ can be solved for x. **True**

(108) The linear combinations $a_1v_1 + a_2v_2$and $b_1v_1 + b_2v_2$ can only be equal if $a_1 = b_1$ and

$a_2 = b_2$. **False**

(109) If each component of a vector in R^3 is doubled, the norm of that vector is doubled. **True**

(110) In R^2 , the vectors of norm 5 whose initial points are at the origin have terminal points lying on a circle of radius 5 centered at the origin. **True**

(111) Every vector in R^n has a positive norm. **False**

(112) If v is a nonzero vector in R^n , there are exactly two unit vectors that are parallel to v. **True**

(113) If $||u|| = 2$, $||v|| = 1$, and $u . v = 1$, then the angle between u and v is $\pi/3$ radians. **True**

(114) The expressions $(u . v) + w$ and $u . (v + w)$ are both meaningful and equal to each other. **False**

(115) If $u . v = u . w$, then $v = w$. **False**

(116) If $u . v = 0$, then either $u = 0$ or $v = 0$. **False**

(117) In R^2 , if u lies in the first quadrant and v lies in the third quadrant, then $u . v$ cannot be positive. **True**

(118) For all vectors $u, v,$ and w in R^n, we have $||u + v + w|| \leq ||u|| + ||v|| + ||w||$ **True**

(119) The vectors $(3, -1, 2)$ and $(0, 0, 0)$ are orthogonal. **True**

(120) If u and v are orthogonal vectors, then for all nonzero scalars k and m, ku and mv are orthogonal vectors. **True**

(121) The orthogonal projection of u along a is perpendicular to the vector component of u orthogonal to a **True**

(122) If a and b are orthogonal vectors, then for every nonzero vector u, we have

$proj_a(proj_h(u) = 0$ **True**

(123) If a and u nonzero vectors , then $proj_a(proj_a(u)) = proj_a(u)$.**True**

(124) the relationship holds for some nonzero vector a, then $u = v$. **False**

(125) For all vectors u and v, it is true that $||u + v|| = ||u|| + ||v||$ **False**

(126) The vector equation of a line can be determined from any point lying on the line and a nonzero vector parallel to the line. True

(127) The vector equation of a plane can be determined from any point lying in the plane and a nonzero vector parallel to the plane. False

(128) The points lying on a line through the origin in $R^2 R^3$ or R3 are all scalar multiples of any nonzero vector on the line. True

(129) All solution vectors of the linear system $Ax = b$ are orthogonal to the row vectors of the matrix A if and only if $b = 0$. True

(130) The general solution of the nonhomogeneous linear system $Ax = b$ can be obtained by adding b to the general solution of the homogeneous linear system $Ax = 0$. False

(131) If x_1 and $x_2$are two solutions of the nonhomogeneous linear system $Ax = b$, then $x_1 - x_2$is a solution of the corresponding homogeneous linear system. True

(132) The cross product of two nonzero vectors u and v is a nonzero vector if and only if u and v are not parallel. True

(133) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane. True

(134) The scalar triple product of $u, v,$ and w determines a vector whose length is equal to the volume of the parallelepiped determined by $u, v,$ and w. False

(135) If u and v are vectors in $3 - space$, then $||v \; x \; u||$is equal to the area of the parallelogram determined by u and v. True

(136) For all vectors $u, v,$ and w in $3 - space$, the vectors $(u \; x \; v) \; x \; w$ and $u \; x \; (v \; x \; w)$ are the same. False

(137) If $u, v,$ and w are vectors in R^3 , where u is nonzero and $u \; x \; v \; = u \; x \; w$, then $v = w$. False

(138) A vector is a directed line segment (an arrow). False

(139) A vector is an $n - tuple$ of real numbers. False

(140) A vector is any element of a vector space. True

(141) There is a vector space consisting of exactly two distinct vectors. False

(142) The set of polynomials with degree exactly 1 is a vector space under the operations defined in Exercise 12. False

(143) Every subspace of a vector space is itself a vector space. True

(144) Every vector space is a subspace of itself. True

(145) Every subset of a vector space V that contains the zero vector in V is a subspace of V. False

(146) The set R^1 is a subspace of R^3 False

(147) The solution set of a consistent linear system $Ax = b$ of m equations in n unknowns is a subspace of R^n. False

(148) The span of any finite set of vectors in a vector space is closed under addition and scalar multiplication. True

(149) The intersection of any two subspaces of a vector space V is a subspace of V. True

(150) The union of any two subspaces of a vector space V is a subspace of V. False

(151) Two subsets of a vector space V that span the same subspace of V must be equal. False

(152) The set of upper triangular $n\ x\ n$ matrices is a subspace of the vector space of all $n\ x\ n$ matrices. True

(153) The polynomials $x - 1$, $(x - 1)^2$, and $(x - 1)^3$ span P_3. False

(154) A set containing a single vector is linearly independent. False

(155) The set of vectors $\{v, kv\}$ is linearly dependent for every scalar k. True

(156) Every linearly dependent set contains the zero vector. False

(157) If the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent, then is also linearly independent for every nonzero scalar k. True

(158) If $v_1, \ldots, v_n$ are linearly dependent nonzero vectors, then at least one vector v_k is a unique linear combination of $v_1, \ldots, v_{k-1}$ True

(159) The set of $2x2$ matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}. False

(160) The three polynomials $(x-1)(x+2)$, $(x+2)$ and $x(x-1)$ are linearly independent. True

(161) The functions f_1 and f_2 are linearly dependent if there is a real number x so that

$k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2. False

(162) If $V = span\{v_1, \ldots, v_n\}$, then $\{v_1, \ldots, v_n\}$ is a basis for V. False

(163) Every linearly independent subset of a vector space V is a basis for V. False

(164) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$ True

(165) The coordinate vector of a vector x in R^n relative to the standard basis for R^n is x. True

(166) Every basis of P_4 contains at least one polynomial of degree 3 or less. False

(167) The zero vector space has dimension zero. True

(168) There is a set of 17 linearly independent vectors in R^{17}. True

(169) There is a set of 11 vectors that span R^{17}. False

(170) Every linearly independent set of five vectors in R^5 is a basis for R^5. True

(171) Every set of five vectors that spans R^5 is a basis for R^5 . True

(172) Every set of vectors that spans R^n contains a basis for R^n. True

(173) Every linearly independent set of vectors in R^n is contained in some basis for R^n . True

(174) There is a basis for M_{22} consisting of invertible matrices. True

(175) If A has size $n \times n$ and In, $A, A^2, \dots, A^{n2}$ are distinct matrices, then $\{I^n, A, A^2, \dots A^{n2}\}$ is linearly dependent. True

(176) There are at least two distinct three-dimensional subspaces of p^2 . False

(177) If B_1 and B_2 are bases for a vector space V, then there exists a transition matrix from B_1 to B_2 . True

(178) Transition matrices are invertible. True

(179) If B is a basis for a vector space R^n, then $P\ B \rightarrow B$ is the identity matrix. True

(180) If $PB_1 \rightarrow B_2$is a diagonal matrix, then each vector in is a scalar multiple of some vector in B_1 . True

(181) If each vector in B_2 is a scalar multiple of some vector in B_1 , then $P\ B_1 \rightarrow B_2$ is a diagonal matrix. False

(182) If A is a square matrix, then $A = PB_1 \rightarrow B_2$ for some bases B_1 andB_2 for R^n . False

(183) The span of $v_1, \dots, v_n$ is the column space of the matrix whose column vectors are $v_1, \dots, v_n$. True

(184) The column space of a matrix A is the set of solutions of $Ax = b$. False

(185) If R is the reduced row echelon form of A, then those column vectors of R that contain the leading 1's form a basis for the column space of A. False

(186) The set of nonzero row vectors of a matrix A is a basis for the row space of A. False

(187) If A and B are $n \times n$ matrices that have the same row space, then A and B have the same column space. False

(188) If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the null space of $E\ A$ is the same as the null space of A. True

(189) If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the row space of $E\ A$ is the same as the row space of A. True

(190) If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the column space of $E\ A$ is the same as the column space of A. False

(191) The system $Ax = b$ is inconsistent if and only if is not in the column space of A. True

(192) There is an invertible matrix A and a singular matrix B such that the row spaces of A and B are the same. False

(193) Either the row vectors or the column vectors of a square matrix are linearly independent. False

(194) A matrix with linearly independent row vectors and linearly independent column vectors is square. True

(195) The nullity of a nonzero $m\ x\ n$ matrix is at most m. False

(196) Adding one additional column to a matrix increases its rank by one. False

(197) The nullity of a square matrix with linearly dependent rows is at least one. True

(198) If A is square and $Ax = b$ is inconsistent for some vector , then the nullity of A is zero. False

(199) If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space. False

(200) If $rank\ (A^T) = rank\ (A)$, then A is square. False

(201) There is no $3x3$ matrix whose row space and null space are both lines in $3 - space$. True

(202) If A is a $2x3$ matrix, then the domain of the transformation TA is R^2 . False

(203) If A is an $m\ x\ n$ matrix, then the codomain of the transformation T_A is R^n . False

(204) If $T: R^n \rightarrow R^m$ and $T(0) = 0$, then T is a matrix transformation. False

(205) If $T: R^n \rightarrow R^m$ and $T(c_1x + c_2y) = c_1T(x) + c_2T(y)$ for all scalars c_1 and c_2 and all vectors x and y in R^n , then T is a matrix transformation. True

(206) There is only one matrix transformation $T: R^n \rightarrow R^m$ such that $T(-x) = -T(x)$ for every vector x in R^n . False

(207) There is only one matrix transformation $T: R^n \rightarrow R^m$ such that $T(x + y) = T(x - y)$ for all vectors x and y in R^n . True

(208) If is a nonzero vector in Rn , then $T(x) = x + b$ is a matrix operator on Rn . False

(209) The matrix $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is the standard matrix for a rotation . False

(210) The standard matrices of the reflections about the coordinate axes in $2 - space$ have the form $\begin{matrix} a & 0 \\ 0 & -a \end{matrix}$, where $a = \pm 1$ True

(211) If $T: R^n \rightarrow R^m$ is a one-to-one matrix transformation, then there are no distinct vectors x and y for which $T(x - y) = 0$ True

(212) If $T: R^n \rightarrow R^m$ is a matrix transformation and $m > n$, then T is one-to-one. False

(213) If $T: R^n \rightarrow R^m$ is a matrix transformation and $m = n$, then T is one-to-one. **False**

(214) If $T: R^n \rightarrow R^m$ is a matrix transformation and $m < n$, then T is one-to-one. **False**

(215) The image of the unit square under a one-to-one matrix operator is a square. **False**

(216) $A\ 2x2$ invertible matrix operator has the geometric effect of a succession of shears, compressions, expansions, and reflections. **True**

(217) The image of a line under a one-to-one matrix operator is a line. **True**

(218) Every reflection operator on R^2 is its own inverse. **True**

(219) The matrix $\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$ represents reflection about a line. **False**

(220) The matrix $\begin{matrix} 1 & -2 \\ 2 & 1 \end{matrix}$ represents a shear. **False**

(221) The matrix $\begin{matrix} 1 & 0 \\ 1 & 3 \end{matrix}$ represents an expansion . **True**

(222) If A is a square matrix and $Ax = \lambda x$ for some nonzero scalar λ , then x is an eigenvector of A. **False**

(223) If λ is an eigenvalue of a matrix A, then the linear system $(\lambda I - A)x = 0$ has only the trivial solution. **False**

(224) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible. **True**

(225) If λ is an eigenvalue of a matrix A, then the eigenspace of A corresponding to λ is the set of eigenvectors of A corresponding to λ . **False**

(226) If 0 is an eigenvalue of a matrix A, then A^2 is singular. **True**

(227) The eigenvalues of a matrix A are the same as the eigenvalues of the reduced row echelon form of A. **False**

(228) If 0 is an eigenvalue of a matrix A, then the set of columns of A is linearly independent. **False**

(229) Every square matrix is similar to itself. **True**

(230) If $A, B,$ and C are matrices for which A is similar to B and B is similar to C, then A is similar to C. **True**

(231) If A and B are similar invertible matrices, then A^{-1} and B^{-1} are similar. **True**

(232) If A is diagonalizable, then there is a unique matrix P such that $P^{-1}AP$ is diagonal. **False**

(233) If A is diagonalizable and invertible, then A^{-1} is diagonalizable. **True**

(234) If A is diagonalizable, then A^T is diagonalizable. **True**

(235) If there is a basis R^n for consisting of eigenvectors of an $n\ x\ n$ matrix A, then A is diagonalizable. **True**

(236) If every eigenvalue of a matrix A has algebraic multiplicity 1, then A is diagonalizable. True

(237) There is a real $5x5$ matrix with no real eigenvalues. False

(238) The eigenvalues of a $2x2$ complex matrix are the solutions of the equation

$\lambda^2 - tr(A)\lambda + det(A) = 0$ True

(239) Matrices that have the same complex eigenvalues with the same algebraic multiplicities have the same trace. False

(240) If λ is a complex eigenvalue of a real matrix A with a corresponding complex eigenvector v, then $\bar{\lambda}$ is a complex eigenvalue of A and $\bar{v}$ is a complex eigenvector of A corresponding to $\bar{\lambda}$. True

(241) Every eigenvalue of a complex symmetric matrix is real. False

(242) If a $2x2$ real matrix A has complex eigenvalues and x_0 is a vector in R^2 , then the vectors $x_0, Ax_0, A^2x_0, \ldots\ldots, A^n x_0$ lie on an ellipse. False

(243) The dot product on R^2 is an example of a weighted inner product. True

(244) The inner product of two vectors cannot be a negative real number. False

(245) $\langle u.v + w\rangle = \langle v.u\rangle + \langle w.u\rangle$ True

(246) $\langle ku.kv\rangle = k2^2\langle u.v\rangle$. True

(247) If $\langle u.v\rangle = 0,$ then u=0 or $v = 0$. False

(248) If $||v||^2 = 0$, then $v = 0$. True

(249) If A is an $n\ x\ n$ matrix, then $\langle u.v\rangle = Au \cdot Av$ defines an inner product on R^n . False

(250) If u is orthogonal to every vector of a subspace W, then $u = 0$. False

(251) If u is a vector in both W and W^1, then $u = 0$. True

(252) If u and v are vectors in $W\perp$, then u+v is in $W\perp$. True

(253) If u is a vector in W^1 and k is a real number, then ku is in W^1. True

(254) If u and v are orthogonal, then $|\langle u.v\rangle| = ||u||||v||$. False

(255) If u and v are orthogonal, then $||u + v|| = ||u|| + ||v||$. False

(256) If A is an $m\ x\ n$ matrix, then A^TA is a square matrix. True

(257) If A^TA is invertible, then A is invertible. False

(258) If A is invertible, then A^TA is invertible. True

(259) If $Ax = b$ is a consistent linear system, then $A^TAx = A^Tb$ is also consistent. True

(260) If $Ax = b$ is an inconsistent linear system, then $A^TAx = A^Tb$ is also inconsistent. False

(261) Every linear system has a least squares solution. True

(262) Every linear system has a unique least squares solution. False

(263) If A is an $m \times n$ matrix with linearly independent columns and b is in R^m , then $Ax = b$ has a unique least squares solution. True

(264) The matrix $\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$ is orthogonal . False

(265) The matrix $\begin{matrix} 1 & -2 \\ 2 & 1 \end{matrix}$ is orthogonal. False

(266) An $m \times n$ matrix A is orthogonal if $A^T A = I$. False

(267) A square matrix whose columns form an orthogonal set is orthogonal. False

(268) Every orthogonal matrix is invertible. True

(269) If A is an orthogonal matrix, then A^2 is orthogonal and $(det\ A)^2 = 1$. True

(270) Every eigenvalue of an orthogonal matrix has absolute value 1. True

(271) If A is a square matrix and $||Au|| = 1$ for all unit vectors u, then A is orthogonal. True

(272) If A is a square matrix, then AA^T and $A^T A$ are orthogonally diagonalizable. True

(273) If v_1 and v_2 are eigenvectors from distinct eigenspaces of a symmetric matrix, then

$||v_1 + v_2||^2 = ||v_1||^2 + ||v_2||^2$. True

(274) Every orthogonal matrix is orthogonally diagonalizable. False

(275) If A is both invertible and orthogonally diagonalizable, then A^{-1} is orthogonally diagonalizable. True

(276) Every eigenvalue of an orthogonal matrix has absolute value 1. True

(277) If A is an $n \times n$ orthogonally diagonalizable matrix, then there exists an orthonormal basis for R^n consisting of eigenvectors of A. False

(278) If A is orthogonally diagonalizable, then A has real eigenvalues. True

(279) A symmetric matrix with positive definite eigenvalues is positive definite. True

(280) $x\frac{2}{1} - \frac{2}{2} - \frac{2}{3} + 4x_1x_2x_3$ is a quadratic form . False

(281) $(x_1 - 3x_2)^2$ is a quadratic form. True

(282) A positive definite matrix is invertible. True

(283) A symmetric matrix is either positive definite, negative definite, or indefinite. False

(284) If A is positive definite, then $-A$ is negative definite. True

(285) $X.X$ is a quadratic form for all x in R^n . True

(286) If $x^T Ax$ is a positive definite quadratic form, then so is $x^T A^{-1} x$. True

(287) If A is a matrix with only positive eigenvalues, then $x^T Ax$ is a positive definite quadratic form. False

(288) If A is a $2x2$ symmetric matrix with positive entries and $det(A) > 0$, then A is positive definite. True

(289) If $x^T Ax$ is a quadratic form with no cross product terms, then A is a diagonal matrix. False

(290) If $x^T Ax$ is a positive definitequadratic form in two variables and $c \neq 0$, then the graph of the equation $x^T Ax = c$ is an ellipse. False

(291) The matrix $\begin{matrix} 0 & i \\ i & 2 \end{matrix}$ is Hermitian . False

(292) The matrix $\begin{matrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ 0 & -\frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{matrix}$ is unitary . False

(293) The conjugate transpose of a unitary matrix is unitary. True

(294) Every unitarily diagonalizable matrix is Hermitian. False

(295) A positive integer power of a skew-Hermitian matrix is skew-Hermitian. False

(296) If $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ for all vectors v_1 and v_2 in V and all scalars c_1 and c_2, then T is a linear transformation. True

(297) If v is a nonzero vector in V, then there is exactly one linear transformation $T: V \rightarrow W$ such that $T(-v) = -T(v)$ False

(298) There is exactly one linear transformation $T: V \rightarrow W$ for which $T(u + v) = T(u - v)$ for all vectors u and v in V. True

(299) If v_0 is a nonzero vector in V, then the formula $T(v) = v_0 + v$ defines a linear operator on V. False

(300) The kernel of a linear transformation is a vector space. True

(301) The range of a linear transformation is a vector space. True

(302) If $T: P_6 \rightarrow M_{22}$ is a linear transformation, then the nullity of T is 3. False

(303) The function $T: M_{22} \rightarrow R$ defined by $T(A) = det\ A$ is a linear transformation. False

(304) The linear transformation $T: M_{22} \rightarrow M_{22}$ defined by $T(A) = \begin{matrix} 1 & 3 \\ 2 & 6 \end{matrix} A$ has rank 1. False

(305) The vector spaces R^2 and P_2 are isomorphic. False

(306) If the kernel of a linear transformation $T: P_3 \to P_3$ is $\{0\}$, then T is an isomorphism. True

(307) Every linear transformation from M_{22} to P_9 is an isomorphism. False

(308) There is a subspace of M_{23} that is isomorphic to R^4. True

(309) There is a $2x2$ matrix P such that $T: M_{22} \to M_{22}$ defined by $T(A) = AP - PA$ is an isomorphism. False

(310) There is a linear transformation $T: P_4 \to P_4$ such that the kernel of T is isomorphic to the range of T False

(311) The composition of two linear transformations is also a linear transformation. True

(312) If $T_1: V \to V$ and $T_2: V \to V$ are any two linear operators, then $T_1 \circ T_2 = T_2 \circ T_1$ False

(313) The inverse of a linear transformation is a linear transformation. False

(314) If a linear transformation T has an inverse, then the kernel of T is the zero subspace. True

(315) If $T: R^2 \to R^2$ is the orthogonal projection onto the x-axis, then $T^{-1}: R^2 \to R^2$ maps each point on the x-axis onto a line that is perpendicular to the x-axis. False

(316) If $T_1: U \to V$ and $T_2: U \to V$ are linear transformations, and if T^1 is not one-to one, then neither is $T_2 \circ T_1$ True

(317) If the matrix of a linear transformation $T: V \to W$ relative to some bases of V and W is $\begin{matrix} 2 & 4 \\ 0 & 3 \end{matrix}$, then there is a nonzero vector x in V such that $T(x) = 2x$. False

(318) If the matrix of a linear transformation $T: V \to W$ relative to bases for V and W is $\begin{matrix} 2 & 4 \\ 0 & 3 \end{matrix}$, then there is a nonzero vector x in V such that $T(x) = 4x$. False

(319) If the matrix of a linear transformation $T: V \to W$ relative to certain bases for V and W is $\begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix}$, then T is one-to-one. True

(320) If $S: V \to V$ and $T: V \to V$ are linear operators and B is a basis for V, then the matrix of $S \circ T$ relative to B is $[T]_B\ [s]_B$ False

(321) If $T: V \to V$ is an invertible linear operator and B is a basis for V, then the matrix for T^{-1} relative to B is $[T]^{-1}{}_B$. True

(322) Every square matrix has an LU-decomposition. False

(323) If a square matrix A is row equivalent to an upper triangular matrix U, then A has an LU-decomposition. False

(324) If $L_1, L_2, \ldots, L_k$ are n x n lower triangular matrices, then the product $L_1, L_2, \ldots, L_k$ is lower triangular. True

(325) If a square matrix A has an LU-decomposition, then A has a unique LDU decomposition. True

(326) Every square matrix has a PLU-decomposition. True

(327) If A is an $m\ x\ n$ matrix, then $A^T A$ is an $m\ x\ m$ matrix False

(328) If A is an $m\ x\ n$ matrix, then $A^T A$ is a symmetric matrix. True

(329) If A is an $m\ x\ n$ matrix, then the eigenvalues of $A^T A$ are positive real numbers. False

(330) If A is an $n\ x\ n$ matrix, then A is orthogonally diagonalizable. False

(331) If A is an $m\ x\ n$ matrix, then $A^T A$ is orthogonally diagonalizable. True

(332) The eigenvalues of $A^T A$ are the singular values of A. False

(333) Every $m\ x\ n$ matrix has a singular value decomposition. True

The End

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